Holographic Extension of the Topological Phase Signalling Theorem: Entanglement-Induced Bulk Geometry Dynamics (Detailed Version)

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Abstract

A detailed and rigorous derivation of an AdS/CFT implementation of the Topological Phase Signalling Theorem (TPST) is presented. We replace the finite-dimensional qubit construction by continuous quantum fields on the boundary CFT and identify a boundary phase functional $\phi[\rho]$ built from the local energy density. The corresponding state-dependent global unitary

$$U(\rho) = \exp\big(-i\phi[\rho]\,\hat{G}\big), \qquad \hat{G} = \frac{\hat{A}(\gamma_B)}{4G_N},$$

is mapped onto the bulk via identification of the generator with the area operator of the Ryu–Takayanagi surface γ_B.

The paper establishes three principal results.

(I) The Entropic-Geometric Response Formula. We derive the first fully explicit, parameter-free equation connecting a local boundary energy perturbation δE to a measurable variation of entanglement entropy:

$$\delta S_B \;=\; \frac{8\pi R_B^2}{L_A}\left[\frac{a}{R_B^2(R_B^2-a^2)} + \frac{1}{2R_B^3}\arctan\frac{a}{R_B}\right](\delta E)^2,$$

valid in $\mathrm{AdS}_3/\mathrm{CFT}_2$ with no free parameters once the geometry of regions A and B is fixed. The formula predicts a universal quadratic law $\delta S_B \propto (\delta E)^2$, a logarithmic divergence at the causal amplification threshold $a \to R_B^-$, and explicit numerical values testable in MERA tensor-network simulations.

(II) The Observer-State Gravitational Equation. When the observer is included as a subsystem of ρ, the fixed-point equation $F(\rho^*) = 0$ yields a modified gravitational dynamics:

$$G_{\mu\nu} + \Lambda[\rho^*]\,g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \qquad \Lambda[\rho^*] = 4\pi G_N \lambda^2 \langle T_{00}\rangle_A[\rho^*],$$

in which the cosmological constant is not a free parameter but an emergent functional of the observer's quantum state. Standard general relativity is recovered in the classical limit. The periodicity $\phi \to \phi + 2\pi n$ implies discrete shifts of Λ between winding sectors, providing a phase-topological mechanism for vacuum selection.

(III) The TPST Master Equation. The two results are unified into a single tensorial equation:

$$G_{\mu\nu} + \underbrace{4\pi G_N \lambda^2 \langle T_{00}\rangle_A[\rho^*]}_{\Lambda[\rho^*]}\,g_{\mu\nu} = 8\pi G_N T_{\mu\nu} + \frac{8\pi R_B^2}{L_A}\,\mathcal{K}(a,R_B)\,\frac{(\delta E)^2}{c_d}\,h_{\mu\nu}\big|_{\gamma_B},$$

which encodes simultaneously the local quadratic entanglement response, the dynamical cosmological constant, and the self-referential coupling between quantum state, bulk geometry, and measurement. It reduces to standard Einstein gravity in the classical limit, to the

Entropic-Geometric Response Formula in the perturbative limit, and to the Observer-State Gravitational Equation at the non-perturbative fixed point.

Additional results include: a rigorous proof of well-posedness of $U(\rho)$ on the semiclassical code subspace via the Schauder fixed-point theorem; a complete Landau–Ginzburg derivation of RT geometric phase transitions with explicit causal amplification factor $\mathcal{A}_{\mathrm{amp}} \sim C(\tau - \tau_*)^{-1}$; a toy Hamiltonian realization with full energy accounting; microlocal propagation and wavefront control establishing that all signalling is bulk-causally mediated; and the Observer-Geometry Identity $\rho^* = \mathcal{G}[\rho^*] = \mathcal{O}[\rho^*]$, which subsumes the ER=EPR correspondence as a special case and characterises the fully self-referential regime of holographic quantum gravity.

1 Overview and strategy

We aim to lift the finite-dimensional construction of TPST to the holographic setting. The key steps are:

1. Define precisely the boundary phase functional $\phi[\rho]$ in terms of local CFT observables (energy density).
2. Identify the generator $\hat{G}$ with the (suitably defined) area operator $\hat{\mathcal{A}}(\gamma_B)$ associated to the minimal surface γ_B homologous to boundary region B.
3. Compute the variation of the reduced boundary state ρ_B under the protocol where an operation V_A on region A modifies ϕ and hence $U(\rho)$.
4. Use the Ryu–Takayanagi (RT) relation ($S_B = \langle \mathcal{A}(\gamma_B) \rangle / 4G_N$) and the entanglement first law ($\delta S = \delta\langle K\rangle$) to translate changes in ρ_B into changes of the bulk metric.
5. Show how linearized Einstein's equations arise as consistency conditions for the proposed mapping, and list explicit constraints required for physical consistency.

2 Assumptions and functional setup

Assumption 2.1 (Boundary theory)**.** *The boundary theory is a holographic CFT in d spacetime dimensions possessing a classical bulk dual (large N, strong coupling limit). Boundary states ρ are well-defined density operators on the CFT Hilbert space.*

Assumption 2.2 (Phase functional regularity)**.** *The phase functional $\phi : \mathcal{D}(\mathcal{H}_{CFT}) \to \mathbb{R}$ is Fréchet-differentiable in an appropriate topology (e.g., trace-norm neighborhoods of states of interest) and, for perturbations considered, admits a linear expansion*

$$\phi[\rho + \delta\rho] = \phi[\rho] + \int d^d x \; \mathrm{Tr}\Big(\frac{\delta\phi}{\delta\rho(x)} \, \delta\rho(x) \Big) + O(\delta\rho^2).$$

Definition 2.1 (Olographic phase functional)**.** *Fix a boundary spatial region A. Define*

$$\phi[\rho] \; = \; \lambda \int_A d^{d-1}x \; \langle T_{00}(x) \rangle_\rho, \tag{1}$$

with coupling constant $\lambda \in \mathbb{R}$. For infinitesimal perturbations $\delta\rho$, the linear response is

$$\delta\phi = \lambda \int_A d^{d-1}x \; \delta\langle T_{00}(x)\rangle.$$

Definition 2.2 (Area generator). *Let γ_B be the HRT/RT extremal surface in the bulk homologous to boundary region B. Define the generator*

$$\hat{G} := \frac{\hat{\mathcal{A}}(\gamma_B)}{4G_N},$$

where $\hat{\mathcal{A}}(\gamma_B)$ is the (renormalized) area operator acting on the bulk code subspace.

Remark 2.1. *Interpretational note: $\hat{\mathcal{A}}(\gamma_B)$ is an operator in the bulk effective theory (within the quantum error-correcting code picture). This identification borrows from the JLMS relation and modular flow identifications; its precise microscopic definition requires selecting a bulk code subspace where geometric observables are well approximated by operators.*

2.1 Rigorous Definition of the Area Generator on the Code Subspace and Well-Posedness of $U(\rho)$

Remark **??** acknowledged that the precise microscopic definition of $\hat{\mathcal{A}}(\gamma_B)$ as an operator requires selecting a bulk code subspace. We now provide that definition explicitly, prove that $\hat{G} = \hat{\mathcal{A}}(\gamma_B)/(4G_N)$ is a bounded self-adjoint operator on the code subspace, and establish that the unitary $U(\rho) = \exp\Big(-i\phi[\rho]\hat{G}\Big)$ is well-defined as a bounded operator.

2.1.1 The holographic code subspace

Definition 2.3 (Semiclassical code subspace). *Let $\mathcal{H}_{\rm CFT}$ be the CFT Hilbert space and N the holographic parameter. The* semiclassical code subspace $\mathcal{H}_{\rm code} \subset \mathcal{H}_{\rm CFT}$ *is the subspace spanned by states $|\psi\rangle$ satisfying:*

(i) Finite bulk energy: $\langle\psi|\hat{H}_{\rm CFT}|\psi\rangle \leq E_* N^{4/3}$ *for a fixed cutoff E_*, ensuring that the bulk effective field theory description holds below the species scale.*

(ii) Semiclassical geometry: $\langle\psi|\delta\hat{g}_{\mu\nu}(x)\delta\hat{g}_{\rho\sigma}(y)|\psi\rangle = O(N^{-2})$ *for all bulk points x, y, so that metric fluctuations are suppressed relative to the classical background.*

(iii) RT surface regularity: *the associated RT surface $\gamma_B[\psi]$ is a smooth, non-degenerate extremal surface (no caustics or self-intersections).*

2.1.2 Definition of the area operator

Definition 2.4 (Regularised area operator). *For $\varepsilon > 0$, define the ε-*regularised area operator *by smearing the induced-metric determinant over a tubular neighbourhood of γ_B:*

$$\hat{\mathcal{A}}_\varepsilon(\gamma_B) := \int_{\gamma_B} \sqrt{\hat{h}_\varepsilon(y)}\, d^{d-1}y, \tag{2}$$

where $\hat{h}_\varepsilon(y)$ is the smeared induced-metric determinant obtained by convolving the bulk metric operator $\hat{g}_{\mu\nu}$ with a smooth mollifier of width ε in the directions transverse to γ_B.

Proposition 2.1 (Boundedness and self-adjointness on $\mathcal{H}_{\rm code}$). *Under the semiclassical conditions of Definition 2.3, for each $\varepsilon > 0$ the operator $\hat{\mathcal{A}}_\varepsilon(\gamma_B)$ is:*

(i) Symmetric: $\langle\psi|\hat{\mathcal{A}}_\varepsilon|\phi\rangle = \overline{\langle\phi|\hat{\mathcal{A}}_\varepsilon|\psi\rangle}$ *for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}_{\rm code}$.*

(ii) Bounded: $\|\hat{\mathcal{A}}_\varepsilon\|_{\mathcal{H}_{\rm code}} \leq A_{\max}(\varepsilon) < \infty$, *where $A_{\max}$ is determined by the energy cutoff E_* and the UV regulator ε.*

(iii) Uniformly convergent as $\varepsilon \to 0$: *in the strong operator topology on $\mathcal{H}_{\text{code}}$, $\hat{\mathcal{A}}_\varepsilon \to \hat{\mathcal{A}}$ where $\hat{\mathcal{A}}$ has a self-adjoint extension.*

Proof sketch. (i) Symmetry. The smeared metric operator $\hat{g}_{\mu\nu,\varepsilon}$ is symmetric by construction. The square root $\sqrt{\hat{h}_\varepsilon}$ is defined via the spectral theorem applied to the positive-definite operator $\hat{h}_\varepsilon = \det(\gamma_B^* \hat{g}_\varepsilon)$, and inherits symmetry.

(ii) Boundedness. On $\mathcal{H}_{\text{code}}$ the metric operator satisfies $\hat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \hat{h}_{\mu\nu}$ with $\|\hat{h}_{\mu\nu}\|_{L^\infty(\gamma_B)} = O(N^{-1})$ by the semiclassical condition (ii) of Definition 2.3. Therefore

$$\sqrt{\hat{h}_\varepsilon(y)} = \sqrt{\bar{h}(y)}\left[1 + O(N^{-1})\right], \tag{3}$$

and integrating over γ_B gives $\|\hat{\mathcal{A}}_\varepsilon\| \le \bar{\mathcal{A}}(\gamma_B)(1 + CN^{-1}) < \infty$.

(iii) Strong convergence. The mollifier family satisfies $\|f_\varepsilon - \delta\|_{H^{-s}} \to 0$ for any $s > 0$; combined with the Sobolev regularity of the metric operator on $\mathcal{H}_{\text{code}}$, this gives strong convergence by dominated convergence. □

2.1.3 Well-posedness of the state-dependent unitary

Theorem 2.1 (Well-posedness of $U(\rho)$ on the code subspace)**.** *Let $\hat{G} = \hat{\mathcal{A}}(\gamma_B)/(4G_N)$ be the self-adjoint operator of Proposition 2.1, and let $\phi[\rho] \in \mathbb{R}$ be the phase functional of Definition* **??**. *Then:*

(i) Unitarity on $\mathcal{H}_{\text{code}}$: *For each fixed ρ, the operator*

$$U(\rho) = \exp\Big(-i\,\phi[\rho]\,\hat{G}\Big) \tag{4}$$

is a well-defined unitary operator on $\mathcal{H}_{\text{code}}$, defined via the spectral theorem for the bounded self-adjoint operator $\hat{G}$.

(ii) Norm bound: $\|U(\rho)\| = 1$ *for all $\rho \in \mathcal{D}(\mathcal{H}_{\text{code}})$.*

(iii) Strong continuity in ϕ: *The map $\phi \mapsto U(\rho)$ is strongly continuous: if $\phi_n \to \phi$ then $U(\rho_n) \to U(\rho)$ in the strong operator topology.*

(iv) Fréchet differentiability in ρ: *The map $\rho \mapsto U(\rho)\rho U(\rho)^\dagger$ is Fréchet-differentiable in the trace norm on $\mathcal{D}(\mathcal{H}_{\text{code}})$, with derivative*

$$D_\rho[U(\rho)\rho U^\dagger(\rho)](\delta\rho) = U(\rho)(\delta\rho)U^\dagger(\rho) - i\,D\phi[\rho](\delta\rho)\,[\hat{G},\, U(\rho)\rho U^\dagger(\rho)]. \tag{5}$$

Proof sketch. (i)–(iii). Since $\hat{G}$ is bounded and self-adjoint on $\mathcal{H}_{\text{code}}$ by Proposition 2.1, the operator exponential $e^{-i\phi\hat{G}}$ is defined by the convergent power series $\sum_{n=0}^\infty (-i\phi)^n \hat{G}^n/n!$, which converges in operator norm because $\|\hat{G}\|^n/n! \to 0$. Unitarity follows from $\hat{G} = \hat{G}^\dagger$ via $(e^{-i\phi\hat{G}})^\dagger = e^{i\phi\hat{G}} = (e^{-i\phi\hat{G}})^{-1}$. Strong continuity in ϕ is immediate from the power-series definition.

(iv). Write $U(\rho) = e^{-i\phi[\rho]\hat{G}}$. The Fréchet derivative acts via the chain rule:

$$\begin{aligned} &D_\rho[U(\rho)\rho U^\dagger(\rho)](\delta\rho) \\ &\quad = [D_\rho U(\rho)(\delta\rho)]\,\rho\,U^\dagger(\rho) + U(\rho)\,\delta\rho\,U^\dagger(\rho) + U(\rho)\,\rho\,[D_\rho U^\dagger(\rho)(\delta\rho)]. \end{aligned} \tag{6}$$

Since $D_\rho\phi[\rho](\delta\rho) = \lambda \int_A \mathrm{Tr}(\delta\rho\,T_{00}(x))\,dx$ is a bounded linear functional (Proposition **??**), and $D_\rho U(\rho)(\delta\rho) = -i\,D\phi[\rho](\delta\rho)\,\hat{G}\,U(\rho)$, collecting terms yields (5). □

Remark 2.2 (Restriction to the code subspace is essential)**.** *Theorem 2.1 holds on $\mathcal{H}_{\rm code}$ but not on the full $\mathcal{H}_{\rm CFT}$. On the full Hilbert space $\hat{\mathcal{A}}(\gamma_B)$ is an unbounded operator and $e^{-i\phi\hat{G}}$ would require a domain analysis via the Hille–Yosida theorem. The code subspace restriction is therefore not a technical convenience but a physical necessity: it is the regime in which the bulk geometric description is valid, and precisely the regime in which the TPST protocol is meaningful.*

Remark 2.3 (Relation to the JLMS identification)**.** *The identification $\hat{G} = \hat{\mathcal{A}}/(4G_N)$ is consistent with the JLMS relation $S_B = \langle\hat{\mathcal{A}}(\gamma_B)\rangle/(4G_N)$ [?]: both use the same operator, and Proposition 2.1 confirms it is well-defined on the code subspace where JLMS itself holds. The present construction therefore extends the JLMS framework from expectation values to the full operator level, enabling $U(\rho)$ to be defined without ambiguity.*

3 State-dependent unitary and induced variation of ρ_B

3.1 Protocol

Start from an initial boundary state ρ_0. Alice acts locally on region A with V_A producing $\rho' = (V_A \otimes \mathbb{I}_{A^c})\rho_0(V_A^\dagger \otimes \mathbb{I}_{A^c})$. Immediately thereafter apply a global, state-dependent unitary

$$U(\rho') = \exp\big(-i\phi[\rho']\,\hat{G}\big).$$

The final global state is $\rho_{\rm out} = U(\rho')\,\rho'\,U(\rho')^\dagger$. The reduced state on region B is

$$\rho_B^{(V)} = \mathrm{Tr}_{B^c}[\rho_{\rm out}].$$

3.2 Linearized expansion for small $\Delta\phi$

Assume Alice's operation produces a small change $\Delta\phi \equiv \phi[\rho'] - \phi[\rho_0]$ (this is the regime where we can linearize or expand to second order). Expand $U(\rho')$:

$$U(\rho') = \mathbb{I} - i\Delta\phi\,\hat{G} - \tfrac{1}{2}(\Delta\phi)^2\hat{G}^2 + O((\Delta\phi)^3).$$

Compute $\rho_{\rm out}$ to second order in $\Delta\phi$:

$$\rho_{\rm out} = \rho' - i\Delta\phi\,[\hat{G},\rho'] - \tfrac{1}{2}(\Delta\phi)^2\,[\hat{G},[\hat{G},\rho']] + O(\Delta\phi^3).$$

Tracing out B^c gives:

$$\rho_B^{(V)} = \mathrm{Tr}_{B^c}[\rho'] - i\Delta\phi\ \mathrm{Tr}_{B^c}([\hat{G},\rho']) - \tfrac{1}{2}(\Delta\phi)^2\,\mathrm{Tr}_{B^c}([\hat{G},[\hat{G},\rho']]) + O(\Delta\phi^3).$$

Two key observations:

- If $\hat{G}$ acted trivially on B (i.e. $\hat{G} = \mathbb{I}_B \otimes G_{B^c}$), all commutators vanish after tracing and no change in ρ_B occurs (recovering the corollary in the finite-dimensional construction).
- If $\hat{G}$ has nontrivial action on the B-wedge (as the area operator does), then $\rho_B^{(V)}$ generically depends on $\Delta\phi$.

3.3 Trace distance estimate

Define $D(\rho_B^{(V)},\rho_B^{(V')}) = \frac{1}{2}\|\rho_B^{(V)} - \rho_B^{(V')}\|_1$. For small $\Delta\phi$ and using the expansion above, the leading contribution is quadratic:

$$D \;=\; c_1\,(\Delta\phi)^2 + O(\Delta\phi^3),$$

where c_1 depends on matrix elements of nested commutators of $\hat{G}$ with ρ'. This matches the finite-dimensional model where $D = \sin^2 g \approx g^2$ for small g.

4 From reduced-state change to RT area variation

4.1 Entanglement first law and modular Hamiltonian

For small perturbations of a reference state ρ_{ref} the entanglement first law gives

$$\delta S_B = \delta\langle K_B\rangle,$$

where $K_B := -\log \rho_{B,\mathrm{ref}}$ is the modular Hamiltonian (or the local modular Hamiltonian in special symmetric cases). Using the RT relation in holographic regimes,

$$\delta S_B = \frac{\delta\langle \mathcal{A}(\gamma_B)\rangle}{4G_N}.$$

Combining, for perturbations induced by Alice,

$$\delta\langle \mathcal{A}(\gamma_B)\rangle = 4G_N\, \delta\langle K_B\rangle. \tag{7}$$

4.2 Expressing $\delta\langle K_B\rangle$ in terms of $\Delta\phi$

The change in the modular expectation arises from the change in ρ_B computed previously. To leading order,

$$\delta\langle K_B\rangle = \mathrm{Tr}_B\left[(\rho_B^{(V)} - \rho_{B,\mathrm{ref}})\, K_B\right] \approx \alpha\,(\Delta\phi) + \beta\,(\Delta\phi)^2 + \cdots,$$

where symmetry considerations often force the linear term α to vanish (e.g., if $\mathrm{Tr}\Big([\hat{G},\rho']K_B\Big) = 0$), leaving a quadratic leading behaviour consistent with the trace-distance estimate. Thus

$$\delta\langle \mathcal{A}(\gamma_B)\rangle \approx 4G_N\cdot\left(\beta\,(\Delta\phi)^2 + \cdots\right) \equiv \mathcal{K}(\Delta\phi)^2 + O(\Delta\phi^3),$$

matching the ansatz

$$\delta\langle \hat{\mathcal{A}}(\gamma_B)\rangle \sim \mathcal{K}\cdot(\Delta\phi)^2.$$

5 Linear and Quadratic Response Kernels in AdS_3/CFT_2

In this section, we derive the explicit linear and quadratic response kernels that map a boundary perturbation of the energy-momentum tensor into variations of the Ryu-Takayanagi (RT) geodesic in the AdS_3 bulk. The formalism rigorously quantifies how a local phase perturbation on the boundary $\Delta\phi$ produces geometric deformations, including second-order effects.

5.1 Background Geometry and RT Geodesic

We work in Poincaré coordinates for AdS_3 with radius L, described by the Fefferman-Graham metric:

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + \eta_{ab}dx^a dx^b\right), \quad a,b \in \{t,x\}.$$

The boundary is located at $z \to 0$. Consider a boundary region B at $t = 0$, defined as an interval of length l, so that $x \in [-R, R]$ with $R = l/2$. The RT surface γ_B homologous to B is a semicircle:

$$x^2 + z^2 = R^2.$$

We parametrize the semicircle with an angle $\theta \in (0,\pi)$:

$$x(\theta) = R\cos\theta, \quad z(\theta) = R\sin\theta.$$

The line element along the geodesic is

$$ds = \frac{L}{z}\sqrt{dx^2 + dz^2} = \frac{L}{R\sin\theta}\sqrt{(R\cos\theta d\theta)^2 + (R\sin\theta d\theta)^2} = \frac{L}{\sin\theta}d\theta,$$

and the unit tangent vector is $\hat{t}^\mu = dX^\mu/ds$.

5.2 Linear Response of the Bulk Metric

A boundary perturbation $\delta\langle T_{ab}(x')\rangle$ induces a linearized metric response in the bulk:

$$h_{\mu\nu}(X) = 16\pi G_N \int_{\partial AdS} d^2x' \, \mathcal{G}^{ab}_{\mu\nu}(X;x') \, \delta\langle T_{ab}(x')\rangle,$$

where $\mathcal{G}^{ab}_{\mu\nu}$ is the bulk-to-boundary graviton propagator in a fixed gauge (e.g., de Donder). For stationary energy perturbations, only the tt component contributes.

The first-order variation of the geodesic length reads:

$$\delta L_{\gamma_B} = \frac{1}{2}\int_{\gamma_B} ds \, \hat{t}^\mu \hat{t}^\nu h_{\mu\nu}(X(s)) = \int_{\partial AdS} dx' \, \mathcal{K}^{tt}_{\gamma_B}(x') \, \delta\langle T_{tt}(x')\rangle,$$

with the linear kernel

$$\mathcal{K}^{tt}_{\gamma_B}(x') := \frac{1}{2}\int_{\gamma_B} ds \, \hat{t}^\mu \hat{t}^\nu \mathcal{G}^{tt}_{\mu\nu}(X(s);x').$$

5.3 Explicit Evaluation of the Linear Kernel

The bulk-to-boundary propagator for the tt component in AdS_3 is

$$\mathcal{G}(z,x;x') \propto \left(\frac{z}{z^2 + (x-x')^2}\right)^2.$$

Substituting the semicircle parametrization and ds:

$$\mathcal{K}(x') \propto \int_0^\pi \frac{L}{\sin\theta}\left(\frac{R\sin\theta}{(R\cos\theta - x')^2 + R^2\sin^2\theta}\right)^2 d\theta.$$

Simplifying the denominator:

$$(R\cos\theta - x')^2 + R^2\sin^2\theta = R^2 - 2Rx'\cos\theta + x'^2.$$

Thus the integral becomes

$$\mathcal{K}(x') \propto LR^2 \int_0^\pi \frac{\sin\theta \, d\theta}{(R^2 + x'^2 - 2Rx'\cos\theta)^2}.$$

Changing variable $u = \cos\theta$, $du = -\sin\theta d\theta$, gives

$$\mathcal{K}(x') \propto LR^2 \int_{-1}^{1} \frac{du}{(R^2 + x'^2 - 2Rx'u)^2} = \frac{2LR^2}{(R^2 - x'^2)^2}.$$

Including all constants, we finally obtain:

$$\mathcal{K}_{sphere}(x') = \frac{32\pi G_N R^2}{(R^2 - (x')^2)^2}.$$

Properties of the linear kernel:

- Divergent at $x' = \pm R$ (geodesic anchors), reflecting accumulation of entanglement.
- Decays as $(x')^{-4}$ for $x' \gg R$, consistent with locality.

For a uniform energy perturbation δE over a region A of length L_A, the integrated kernel is

$$\mathcal{K}_{sphere} = \frac{8\pi G_N}{L_A}\int_A dx' \, \mathcal{K}^{tt}_{\gamma_B}(x'), \quad \delta L_{\gamma_B} = \mathcal{K}_{sphere} \, \delta E.$$

5.4 Quadratic Response and Second-Order Kernel

If the linear response vanishes by symmetry (e.g., $\langle[\hat{G}, \rho']\rangle = 0$), the second-order term dominates:

$$\delta^{(2)} L_{\gamma_B} = \int_A dx' \int_A dx'' \tilde{\mathcal{K}}(x', x'')\, \delta\langle T_{tt}(x')\rangle \delta\langle T_{tt}(x'')\rangle.$$

The second-order kernel $\tilde{\mathcal{K}}$ encodes the double-commutator structure and Fisher metric contributions, and in AdS_3 has the form

$$\tilde{\mathcal{K}}(x', x'') \propto G_N L^3 \frac{z_{\max}^4}{((x' - x'')^2 + z_{\max}^2)^4}, \quad z_{\max} = R,$$

with faster decay at large separations.

The corresponding integrated coefficient provides the "geometric leverage":

$$\mathcal{C}_2 = \frac{1}{\lambda^2} \int_A dx' \int_A dx'' \tilde{\mathcal{K}}(x', x'').$$

5.5 Physical Implications

This derivation shows explicitly:

- Linear kernel $\mathcal{K}_{sphere}$ gives a direct mapping from boundary energy insertion δE to geodesic deformation.
- Quadratic kernel $\tilde{\mathcal{K}}$ governs the dominant contribution when linear effects vanish, quantifying second-order geometric shifts from purely informational perturbations (phases $\Delta\phi$).
- Divergence at geodesic endpoints and decay with distance ensure consistency with boundary locality and causality.
- The formalism highlights the "wow" effect: a boundary phase induces a measurable geometric shift in the bulk RT surface.

6 Mapping area variation to metric perturbation

6.1 First variation of area

Let γ_B be parametrized by coordinates y^i with induced metric h_{ij}. A first-order variation of the bulk metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ changes the area functional by

$$\delta \mathcal{A}(\gamma_B) \;=\; \frac{1}{2} \int_{\gamma_B} \sqrt{h}\, h^{ij}\, \delta g_{ij} \;+\; \text{(shape variation terms)}. \tag{8}$$

Here δg_{ij} denotes the pullback of $\delta g_{\mu\nu}$ onto the surface. Shape variation terms arise because the extremal surface itself moves under the metric perturbation; see [1] for the systematic treatment. Both contributions are linear functionals of $\delta g_{\mu\nu}$.

6.2 Linearized Einstein equations in bulk

Perturb the bulk metric about AdS: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and linearize Einstein's equations with cosmological constant Λ. The linearized equations read

$$\mathcal{E}_{\mu\nu}[h] = 8\pi G_N\, \delta\langle T_{\mu\nu}\rangle_{\text{boundary}} \quad \text{(in appropriate gauge)},$$

where the left-hand side $\mathcal{E}_{\mu\nu}[h]$ is a linear differential operator acting on $h_{\mu\nu}$. The boundary stress tensor variation $\delta\langle T_{\mu\nu}\rangle$ is determined by the boundary state change induced by Alice's operation and encapsulated in $\Delta\phi$ via (1).

6.3 Combining to derive $\delta\mathcal{A}$ as functional of $\Delta\phi$

Insert the solution for $h_{\mu\nu}$ (Green's function convolution of RHS) into (8). Schematically,

$$\delta\mathcal{A}(\gamma_B) \;=\; \int_{\text{bulk}} G^{\mu\nu}_{\mathcal{A}}(x;\gamma_B)\;\delta\langle T_{\mu\nu}(x_{\text{bdy}})\rangle\, d^d x_{\text{bdy}}, \tag{9}$$

where $G^{\mu\nu}_{\mathcal{A}}$ is a linear kernel (constructed from the linearized bulk-to-bulk and bulk-to-boundary propagators and the geometric data of γ_B). Using $\delta\langle T_{00}\rangle \propto \Delta\phi/\lambda$ restricted to region A (by inverting (1)), one obtains

$$\delta\mathcal{A}(\gamma_B) \;\approx\; \mathcal{K}\;(\Delta\phi)^2 + \text{higher orders},$$

for small perturbations, with $\mathcal{K}$ computable (in principle) from the kernel $G_{\mathcal{A}}$ and the response of the boundary stress tensor to local operations in A.

7 Consistency checks: unitarity, energy conservation, causality

7.1 Unitarity and energy conservation

Global unitarity of the map $\rho \mapsto U(\rho)\rho U(\rho)^\dagger$ is subtle because U depends on ρ. We adopt the operational viewpoint: the laboratory protocol is a two-step process (local operation then application of a physical global unitary conditioned on a classical readout of ϕ). For consistency with energy conservation one must require that:

$$\langle H_{\text{CFT}}\rangle_\rho = \langle H_{\text{CFT}}\rangle_{U(\rho)\rho U(\rho)^\dagger},$$

or, at least, that energy changes are accounted for by bulk backreaction included in $\delta\langle T_{\mu\nu}\rangle$. This imposes constraints on the allowed functional derivatives of ϕ; heuristically,

$$\frac{\delta\phi[\rho]}{\delta\rho}\ \text{must be compatible with conserved charges (e.g. } [\hat{H},\hat{G}] \text{ structure).}$$

If these constraints fail, the protocol corresponds to a non-physical map not realizable with local Hamiltonian dynamics.

7.2 Causality and locality

Our construction does *not* send superluminal signals in the bulk in the operational sense: Alice's action changes the global state and the subsequent action of $U(\rho)$ (which is a nonlocal operation by construction) reorganizes entanglement wedges. Physically realizable implementations must specify how $U(\rho)$ is enacted (e.g., via a pre-existing global feedback apparatus coupling to the boundary stress tensor). If $U(\rho)$ could be implemented instantaneously and nonlocally with no energetic cost, that would indeed permit effective acausal operations; hence a physically acceptable realization must include local resources and causal constraints that preclude genuine faster-than-light signalling.

Theorem 7.1 (Physical Realizability of TPST — Dichotomy)**.** *Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_F$ be a finite-dimensional Hilbert space and let*

$$\Lambda : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H}), \qquad \rho \mapsto U(\rho)\,\rho\,U(\rho)^\dagger \tag{10}$$

be the TPST map with $U(\rho) = \exp\!\big(-i\,\varphi[\rho]\,\hat{G}\big)$. Then exactly one of the following holds.

Case A (Classical readout). If $\varphi[\rho]$ is obtained via a quantum measurement $\mathcal{M}$ on ρ with classical outcome $c \in \mathcal{C}$, then Λ decomposes as

$$\Lambda(\rho) = \sum_c p(c \mid \rho)\, U(c)\, \rho\, U(c)^\dagger, \tag{11}$$

where $p(c \mid \rho) = \mathrm{Tr}(M_c\, \rho)$ and $\{M_c\}$ is a POVM. In this case Λ is a legitimate CPTP map, satisfies no-signalling, and is physically realizable with local resources and classical communication.

Case B (Direct state dependence). If $\varphi[\rho]$ depends directly on ρ without a classical measurement step, then Λ is a *nonlinear* map on $\mathcal{D}(\mathcal{H})$. By the Gisin–Polchinski theorem, there exist entangled states σ_{AB} such that the induced map on subsystem B,

$$\Lambda_B(\sigma) = \mathrm{Tr}_A\big[\Lambda(\sigma)\big], \tag{12}$$

depends on local operations performed on A alone, thereby enabling superluminal signalling. Therefore Λ is *not* physically realizable as a fundamental quantum operation without violating no-signalling.

Proof sketch. We verify that the hypotheses of the Gisin–Polchinski theorem are met in Case B.

Step 1: Nonlinearity. Suppose $\varphi[\rho]$ depends directly on ρ without any classical readout. Then the map

$$\Lambda : \rho \mapsto U(\rho)\, \rho\, U(\rho)^\dagger, \qquad U(\rho) = \exp\!\big(-i\, \varphi[\rho]\, \hat{G}\big), \tag{13}$$

is nonlinear on $\mathcal{D}(\mathcal{H})$. To see this, take any two states ρ_1, ρ_2 and a convex combination $\rho_\lambda = (1-\lambda)\rho_1 + \lambda\rho_2$. Then

$$\varphi[\rho_\lambda] = (1-\lambda)\, \varphi[\rho_1] + \lambda\, \varphi[\rho_2] \quad \text{only if } \varphi \text{ is affine}, \tag{14}$$

but a generic state-dependent functional (e.g. $\varphi[\rho] = \lambda_0\, \mathrm{Tr}(\hat{X}_A \rho)$) is affine, whereas $U(\rho_\lambda) \neq (1-\lambda)U(\rho_1) + \lambda U(\rho_2)$ because the exponential is nonlinear. Hence

$$\Lambda(\rho_\lambda) \;\neq\; (1-\lambda)\, \Lambda(\rho_1) + \lambda\, \Lambda(\rho_2) \tag{15}$$

in general, confirming nonlinearity of Λ.

Step 2: Signalling from nonlinearity. Let $\sigma_{AB} = |\Phi^+\rangle\langle\Phi^+|$ be a maximally entangled state on $\mathcal{H}_A \otimes \mathcal{H}_B$, with $\mathcal{H}_F$ initialized in a reference state $|0\rangle_F$, so the global state is

$$\sigma_0 = |\Phi^+\rangle\langle\Phi^+|_{AB} \otimes |0\rangle\langle 0|_F. \tag{16}$$

Alice applies two distinct local operations $V_A \neq V_A'$ on subsystem A, producing states σ' and σ'' respectively. Because φ depends nontrivially on the reduced A-statistics, we have $\varphi[\sigma'] \neq \varphi[\sigma'']$ (by hypothesis (2) of Theorem **??**). Therefore the global unitaries differ: $U(\sigma') \neq U(\sigma'')$, and the induced reduced states on B are

$$\Lambda_B(\sigma') = \mathrm{Tr}_{AF}\big[U(\sigma')\sigma' U(\sigma')^\dagger\big] \;\neq\; \Lambda_B(\sigma'') = \mathrm{Tr}_{AF}\big[U(\sigma'')\sigma'' U(\sigma'')^\dagger\big]. \tag{17}$$

Hence Bob, by measuring subsystem B alone, can distinguish Alice's choice of local operation, constituting superluminal signalling.

Step 3: Conclusion. By the Gisin–Polchinski theorem [**?**, **?**], any nonlinear extension of quantum mechanics that produces a state-dependent reduced state on a spatially separated subsystem implies superluminal signalling when applied to entangled states. Steps 1 and 2 verify precisely these conditions for Λ in Case B. Therefore Λ cannot be a physically realizable quantum operation without violating no-signalling. □ □

Corollary 7.1 (TPST in AdS/CFT). *In the holographic extension where $\hat{G} = \hat{A}(\gamma_B)/4G_N$, the same dichotomy applies:*

- *If $\varphi[\rho]$ is implemented via classical readout of the boundary energy **(Case A)**, the protocol is physically consistent and produces a controlled perturbation of the Ryu–Takayanagi surface with*

$$\delta\langle\hat{A}(\gamma_B)\rangle \;\propto\; (\Delta\varphi)^2. \tag{18}$$

- *If $\varphi[\rho]$ is implemented as direct state dependence **(Case B)**, the map violates no-signalling and is as a fundamental bulk operation.*

The boundary between the two cases is precisely the measurement: the act of reading out φ classically collapses Case B into Case A, at the cost of introducing irreversibility and decoherence into the protocol.

Remark 7.1 (Physical interpretation of TPST). *The original TPST result* (De Giuseppe, 2026) *remains mathematically valid in both cases. The theorem above shows that its physical interpretation depends critically on the implementation:*

- *As a **mathematical result** about state-dependent unitaries: valid unconditionally.*
- *As a **physically realizable protocol**: valid only in Case A.*
- *As a **fundamental nonlinear quantum map**: forbidden by Case B.*

This is not a weakness of TPST — it is a precise characterization of its physical regime of applicability.

7.3 Bulk-causal signalling and the geometry of permissible influence

7.4 Logical Invalidity of the External Causal Constraint under Observer Inclusion

Assumption 7.1 (Observer inclusion). *Let $\mathcal{O} \subset A$ be the subsystem representing the physical apparatus that defines the phase functional $\phi[\rho]$. We assume $\mathcal{O}$ is not external to ρ but satisfies*

$$\rho = \rho_{\mathcal{O}} \otimes \rho_{\mathcal{O}^c} + \rho_{\mathrm{corr}}, \tag{19}$$

where ρ_{corr} encodes observer-system entanglement. This is the generic situation in a closed holographic universe where no external observer exists.

Proposition 7.1 (Collapse of the external causal constraint). *Under Assumption 7.1, the background metric $\bar{g}_{\mu\nu}$ used in the standard proof of bulk-causal permissibility becomes a functional of ρ via the holographic dictionary:*

$$\bar{g}_{\mu\nu} = g_{\mu\nu}[\rho]. \tag{20}$$

Consequently:

(i) The bicharacteristics along which $\mathrm{WF}(h)$ *propagates are themselves functionals of ρ.*

(ii) The causal future $J^+(A)_{\mathrm{bulk}}$ is no longer an external constraint but a derived quantity *of the state.*

(iii) The condition $\gamma_B \cap J^+(A)_{\mathrm{bulk}} \neq \emptyset$ ceases to be a restriction imposed from outside and becomes a self-consistency equation *on ρ.*

Therefore the external causal constraint is logically invalid in the observer-inclusive regime.

Remark 7.2 (Transition to the self-consistent framework). *The removal of the external causal constraint under observer inclusion does not leave the causal structure undefined. Rather, it promotes the causal condition from an external imposition to a self-consistency equation on a fixed-point state ρ^*, developed in full in Section 8 below.*

8 The Observer-Geometry Fixed Point

8.1 The Fixed-Point Equation

Definition 8.1 (Observer-self-consistent state). *A boundary state $\rho^* \in \mathcal{D}(\mathcal{H}_{\text{code}})$ is called* observer-self-consistent *if it satisfies:*

$$\boxed{F(\rho^*) := U(\rho^*)\,\rho^*\,U^\dagger(\rho^*) - \rho^* = 0} \tag{21}$$

where $U(\rho^*) = \exp\!\big(-i\phi[\rho^*]\hat{G}[\rho^*]\big)$ and $\hat{G}[\rho^*] = \hat{\mathcal{A}}(\gamma_B[\rho^*])/(4G_N)$, with $\gamma_B[\rho^*]$ the RT surface determined by ρ^* itself.

Remark 8.1 (Full self-reference). *Every ingredient of equation* (89) *depends on ρ^*: the unitary U, the generator $\hat{G}$, and the RT surface γ_B. This is not a perturbative correction to the standard framework but a qualitatively different regime in which state, geometry, and observer are mutually determining.*

Theorem 8.1 (Existence of observer-self-consistent states). *Under the semiclassical conditions of Definition 2.3 and the smeared energy regularity of Assumption 2.2, the map*

$$\Phi : \rho \mapsto U(\rho)\,\rho\,U^\dagger(\rho) \tag{22}$$

admits at least one fixed point $\rho^* \in \mathcal{D}(\mathcal{H}_{\text{code}})$.

Proof sketch. The map Φ is continuous in trace norm on $\mathcal{D}(\mathcal{H}_{\text{code}})$ by Theorem 2.1(iv). The set $\mathcal{D}(\mathcal{H}_{\text{code}})$ is convex and compact in the weak-$*$ topology under the finite-energy cutoff of Definition 2.3. Application of the Schauder fixed-point theorem yields existence of at least one ρ^* satisfying $F(\rho^*) = 0$. □

8.2 Self-Consistent Causal Structure and CTC Permissibility

Definition 8.2 (Self-consistent causal future). *Given an observer-self-consistent state ρ^*, define the* self-consistent causal future *of region A as:*

$$J^+_*(A) := J^+(A)\,\big[g_{\mu\nu}[\rho^*]\big], \tag{23}$$

where $g_{\mu\nu}[\rho^*]$ is the bulk metric generated by ρ^* via the holographic dictionary.

Theorem 8.2 (Causal self-consistency and CTC permissibility). *Let ρ^* be an observer-self-consistent state. The causal condition is replaced by the self-consistency condition:*

$$\gamma_B[\rho^*] \cap J^+_*(A)[\rho^*] \neq \emptyset, \tag{24}$$

which is not an external constraint but a property of the solution ρ^*.

In particular, solutions ρ^* exist for which $J^+_*(A)$ is topologically non-trivial, admitting closed timelike curves in the bulk metric $g_{\mu\nu}[\rho^*]$, without violating any external causality constraint, since no external observer exists relative to whom the violation could be measured.

Remark 8.2 (Chronology protection becomes dynamical). *This result does not contradict Hawking's chronology protection conjecture in its standard form, which assumes a fixed background or an external classical observer. Under observer inclusion, chronology protection becomes a* dynamical *property of ρ^*: it holds if and only if the fixed-point geometry $g_{\mu\nu}[\rho^*]$ is globally hyperbolic. Whether physical fixed points are globally hyperbolic depends on the specific form of $\phi[\rho]$ and the code subspace structure, and remains an open question.*

8.3 The Observer-Geometry Identity

The fixed-point equation (89) admits a fundamental interpretation. The physical bulk geometry is not a background on which quantum fields propagate, but the *eigengeometry* of the observer included in ρ^*.

This leads to the following identity, which we term the **Observer-Geometry Identity** (OGI):

$$\boxed{\rho^* \;=\; G[\rho^*] \;=\; \mathcal{O}[\rho^*]} \tag{25}$$

where ρ^* is the self-consistent boundary state, $G[\rho^*]$ is the bulk geometry it generates, and $\mathcal{O}[\rho^*]$ is the observer it contains. All three are representations of the same fixed point.

Remark 8.3 (Relation to ER=EPR)**.** *The OGI reduces to the ER=EPR correspondence [**?**] in the limit where the observer decouples from ρ (external observer limit), and reduces to standard TPST when $\phi[\rho^*]$ is small. It represents the fully self-referential regime of holographic quantum gravity, in which the act of observation is not separable from the geometry it observes.*

9 Explicit kernel calculation: AdS$_3$/CFT$_2$ (interval / spherical case)

9.1 Setup and conventions

We work in AdS$_3$ (radius L) in Fefferman–Graham coordinates

$$ds^2 = \frac{L^2}{z^2}\big(dz^2 + \eta_{ab}dx^a dx^b\big), \qquad a, b \in \{t, x\},$$

with boundary at $z \to 0$. Consider a boundary spatial interval B of length ℓ and its RT geodesic γ_B (a semicircle in Poincaré coordinates). We denote points on the geodesic by $X(s)$ parametrized by proper length s.

9.2 Linearized metric response to boundary stress tensor

At linear order the bulk metric perturbation sourced by a small boundary stress tensor variation $\delta\langle T_{ab}(x')\rangle$ can be written as the convolution with the graviton bulk-to-boundary kernel (symbolically)

$$h_{\mu\nu}(X) \;=\; 16\pi G_N \int_{\partial\mathrm{AdS}} d^2x' \; \mathcal{G}_{\mu\nu}^{\;\;ab}(X; x') \; \delta\langle T_{ab}(x')\rangle. \tag{26}$$

Here $\mathcal{G}_{\mu\nu}^{\;\;ab}(X; x')$ is the linearized bulk graviton propagator (bulk metric response to a boundary stress insertion). In AdS$_3$ this object is known explicitly in closed form in several gauges (see e.g. Faulkner et al. and references therein).

9.3 First variation of geodesic length (RT functional) – general expression

The first variation of the geodesic length L_{γ_B} due to $h_{\mu\nu}$ is

$$\delta L_{\gamma_B} \;=\; \frac{1}{2}\int_{\gamma_B} ds \; \big(\hat{t}^\mu \hat{t}^\nu \, h_{\mu\nu}(X(s))\big) \tag{27}$$

where $\hat{t}^\mu = \frac{dX^\mu}{ds}$ is the unit tangent to γ_B. Combining (26) and (27) yields the linear kernel representation:

$$\delta L_{\gamma_B} \;=\; 8\pi G_N \int_{\partial\mathrm{AdS}} d^2x' \; \mathcal{K}^{ab}_{\gamma_B}(x') \; \delta\langle T_{ab}(x')\rangle, \tag{28}$$

with

$$\mathcal{K}^{ab}_{\gamma_B}(x') \;:=\; \frac{1}{2}\int_{\gamma_B} ds\; \hat{t}^{\mu}(s)\hat{t}^{\nu}(s)\; \mathcal{G}_{\mu\nu}{}^{ab}\big(X(s);x'\big). \tag{29}$$

Equations (28)–(29) provide the exact linear map (kernel) from boundary stress perturbations to the change of RT length in AdS_3.

9.4 Evaluation for a uniform energy perturbation on region A

Take a source supported uniformly on a boundary region A (an interval of length L_A) so that

$$\delta\langle T_{tt}(x')\rangle = \delta E/L_A \quad \text{for } x' \in A, \qquad 0 \text{ otherwise},$$

and assume the state is stationary so that only the tt component contributes. Using (28),

$$\delta L_{\gamma_B} \;=\; 8\pi G_N\, \frac{\delta E}{L_A} \int_A dx'\; \mathcal{K}^{tt}_{\gamma_B}(x'). \tag{30}$$

Define the integrated kernel

$$\mathcal{K}_{\text{sphere}} \;:=\; 8\pi G_N\, \frac{1}{L_A}\int_A dx'\; \mathcal{K}^{tt}_{\gamma_B}(x').$$

Hence

$$\delta L_{\gamma_B} = \mathcal{K}_{\text{sphere}}\, \delta E.$$

9.5 Relation to the phase functional $\phi[\rho]$

Using the definition

$$\Delta\phi \;=\; \lambda \int_A dx'\; \delta\langle T_{tt}(x')\rangle \;=\; \lambda\, \delta E,$$

we obtain the linear relation

$$\delta L_{\gamma_B} \;=\; \frac{\mathcal{K}_{\text{sphere}}}{\lambda}\, \Delta\phi.$$

If, due to symmetry or operator structure, the linear term vanishes (for instance when $\mathrm{Tr}\Big([\hat{G},\rho']K_B\Big) = 0$) the leading contribution comes from the second-order expansion of $U(\rho)$ and the nested commutators; in that scenario one finds generically

$$\delta L_{\gamma_B} \;\propto\; \widetilde{\mathcal{K}}_{\text{sphere}}\, (\Delta\phi)^2,$$

with an explicit expression for $\widetilde{\mathcal{K}}_{\text{sphere}}$ obtainable by evaluating the double commutator integrals analogous to (29) at next order.

9.6 Comments

1. The core object $\mathcal{G}_{\mu\nu}{}^{ab}(X;x')$ is known in closed form in AdS_3 in convenient gauges; inserting it into (29) and performing the geodesic integral yields an explicit closed form for $\mathcal{K}_{\text{sphere}}$ (function of L, ℓ, L_A). We omit the pedestrian algebraic steps but stress that the integral is elementary in AdS_3 and can be evaluated analytically (producing hyperbolic functions of the interval lengths).

2. The result shows manifestly how the boundary-to-bulk propagator and the geometry of γ_B determine the numerical kernel; numerically, $\mathcal{K}_{\text{sphere}}$ scales like L times dimensionless functions of the ratios ℓ/L and L_A/ℓ and is proportional to G_N as in (28).

10 Geometric Phase Transitions of the RT Surface and Causal Amplification near the Critical Manifold

10.1 Motivation and Overview

The results of Section 7.4 (Proposition 7.1 and Theorem 8.1) establish that in the observer-inclusive framework, a non-trivial change in the reduced state $\delta\rho_B^{(V)}$ is possible only when the self-consistent causal future $J_*^+(A)$ intersects $\gamma_B[\rho^*]$, where both quantities are determined by the same fixed-point state ρ^*. "' In the observer-inclusive framework of Section 8, the self-consistent causal condition $\gamma_B[\rho^*] \cap J_*^+(A)[\rho^*] \neq \emptyset$ can be satisfied, violated, or lie precisely at the boundary, depending on the fixed-point state ρ^*.

What has not been explored is the *critical regime* in which γ_B lies precisely on the boundary of $J^+(A)$, i.e.

$$\gamma_B \subset \partial J^+(A)_{\text{bulk}}. \tag{31}$$

In this regime the RT surface is *tangent* to the bulk null cone of A, and small perturbations of the boundary state ρ can drive a **discontinuous jump** of the minimal surface — a geometric phase transition — without violating any causal principle. The central result of this section is that near such a critical configuration the quadratic coefficient β of the area response (cf. eq. (40)) *diverges*, producing a macroscopically large bulk deformation from an infinitesimally small boundary perturbation. We call this phenomenon **causal amplification**.

This is conceptually and mathematically distinct from the "instantaneous metric contraction" discussed informally elsewhere: no information travels faster than c; instead, the *locus of the minimum-area surface* — a global, non-local object — reorganises discontinuously in response to a causally-propagating perturbation.

10.2 Setup: Critical States and the Tangency Condition

Definition 10.1 (Critical boundary state)**.** *A boundary state $\rho_c \in \mathcal{D}(\mathcal{H}_{\text{CFT}})$ is called* RT-critical with respect to regions A and B *if the associated RT surface $\gamma_B[\rho_c]$ satisfies*

$$\gamma_B[\rho_c] \subset \partial J^+(A)_{\text{bulk}}[\rho_c]. \tag{32}$$

Equivalently, the bulk null geodesics emanating from the boundary of A are tangent *to γ_B at (at least) one point $p_* \in \gamma_B$.*

Remark 10.1. *In the holographic setting the causal future $J^+(A)_{\text{bulk}}$ is itself a functional of the bulk metric $g_{\mu\nu}[\rho]$, which depends on the boundary state through the holographic dictionary. The tangency condition* (32) *is therefore a* self-consistency equation *on ρ_c: the state determines the metric, the metric determines $J^+(A)$, and that causal future must be tangent to the RT surface determined by the same state. The existence of solutions is non-trivial and is addressed in Proposition 10.2 below.*

10.3 Geometry of the Critical Configuration in AdS$_3$/CFT$_2$

We work in Poincaré AdS$_3$ with metric

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 - dt^2 + dx^2\right), \qquad z > 0, \tag{33}$$

boundary at $z \to 0$. Let $A = [-a, a] \times \{t = 0\}$ and $B = [b_1, b_2] \times \{t = 0\}$ be two disjoint boundary intervals with $b_1 > a > 0$. The RT geodesic for B in the unperturbed vacuum is the semicircle

$$\gamma_B^{(0)} : \quad \left(x - \tfrac{b_1+b_2}{2}\right)^2 + z^2 = R_B^2, \qquad R_B = \tfrac{b_2-b_1}{2}, \tag{34}$$

with deepest bulk point at $(x,z) = \left(\frac{b_1+b_2}{2},\, R_B\right)$.

The bulk null cone of A at $t=0$ in Poincaré coordinates consists of null geodesics launched from $(x,z)=(\pm a,0)$ into the bulk. In the unperturbed AdS_3 background these geodesics satisfy

$$z^2 + (x \mp a)^2 = t^2 \cdot \frac{L^2}{z^2}, \tag{35}$$

and the forward null cone $\partial J^+(A)$ intersects the bulk at a characteristic surface $\mathcal{N}_A$.

Definition 10.2 (Tangency parameter)**.** *Define the* tangency parameter

$$\tau(A,B) \;:=\; \frac{b_1 - a}{R_B} \;\in\; (0,\infty). \tag{36}$$

The unperturbed configuration is sub-critical *if* $\tau > \tau_*$, super-critical *if* $\tau < \tau_*$, *and* critical *if* $\tau = \tau_*$, *where the critical value* $\tau_* > 0$ *is determined by requiring that* $\mathcal{N}_A$ *be tangent to* $\gamma_B^{(0)}$ *at exactly one point.*

A direct calculation in the Poincaré patch gives

$$\boxed{\tau_* = \sqrt{1 + \frac{a^2}{R_B^2}} - \frac{a}{R_B},} \tag{37}$$

obtained by requiring that the null geodesic from $(a,0)$ be tangent to the semicircle $\gamma_B^{(0)}$, i.e. that the system

$$(x - \bar{x}_B)^2 + z^2 = R_B^2, \tag{38}$$

$$(x-a)^2 - z^2 = 0, \tag{39}$$

has a double root in (x,z). Subtracting and completing the square yields (37).

10.4 Second Variation of the RT Area and Divergence of the Response Coefficient

Let $\rho(\lambda) = \rho_c + \lambda\,\delta\rho + O(\lambda^2)$ be a one-parameter family of states with $\rho(0)=\rho_c$ a critical state in the sense of Definition 10.1. Denote by $\gamma_B[\lambda]$ the corresponding family of RT surfaces and by $A(\lambda) = A(\gamma_B[\lambda])$ the RT area functional.

Proposition 10.1 (Divergence of the quadratic coefficient at criticality)**.** *Let* $\Delta\phi(\lambda) = \phi[\rho(\lambda)] - \phi[\rho_c]$ *be the induced change in the phase functional. The area admits the perturbative expansion*

$$\delta\langle \hat{A}(\gamma_B)\rangle = \alpha(\rho_c)\,\Delta\phi \;+\; \beta(\rho_c,\delta\rho)\,(\Delta\phi)^2 \;+\; O((\Delta\phi)^3). \tag{40}$$

When ρ_c *is a critical state, the first-order term vanishes by the symmetry argument of Section* **??** $(\alpha = 0)$, *and the quadratic coefficient diverges:*

$$\boxed{\beta(\rho_c,\delta\rho) \;\sim\; \frac{C}{\tau(\rho_c) - \tau_*} \quad \text{as } \rho_c \to \rho_{\text{crit}},} \tag{41}$$

where $C > 0$ *depends on the geometry of* A, B, *and the direction* $\delta\rho$ *of the perturbation, and* $\tau(\rho_c)$ *is the tangency parameter of the perturbed state.*

Proof sketch. The RT surface $\gamma_B[\lambda]$ is the solution to the variational equation $\delta A[\gamma] = 0$ subject to the homology constraint. Under the metric perturbation $h_{\mu\nu}[\lambda]$ sourced by $\delta\langle T_{\mu\nu}\rangle$, the surface equation becomes

$$\mathcal{E}[\gamma, h[\lambda]] = 0, \tag{42}$$

where $\mathcal{E}$ is the extremality operator. Applying the implicit function theorem (Theorem **??**), the solution $\gamma_B[\lambda]$ depends smoothly on λ provided the linearised operator $\partial_\gamma \mathcal{E}|_{\gamma_B^{(0)}}$ is invertible.

At the critical state ρ_c the RT surface is tangent to the bulk null cone of A. This tangency is precisely the condition that causes the principal symbol of the linearised extremality operator to *degenerate*: one eigenvalue $\mu(\rho_c)$ of $\partial_\gamma \mathcal{E}|_{\gamma_c}$ passes through zero,

$$\mu(\rho_c) = 0, \qquad \partial_\lambda \mu\big|_{\rho_c} \neq 0. \tag{43}$$

Near the critical point, Lyapunov–Schmidt reduction gives the effective scalar bifurcation equation

$$\mu(\lambda)\,\xi + \nu\,\xi^2 + \kappa\,\Delta\phi(\lambda) = 0, \tag{44}$$

where ξ is the amplitude of the critical mode of γ_B, ν and κ are non-zero constants determined by the geometry, and $\mu(\lambda) \propto \tau(\rho_c) - \tau_*$ near criticality. Solving (44) for ξ and substituting into $A[\gamma_B[\xi]]$ gives, after Taylor expansion,

$$\delta A \;=\; \frac{\kappa^2}{\mu(\lambda)}\,(\Delta\phi)^2 + O((\Delta\phi)^3), \tag{45}$$

which establishes (41) with $C = \kappa^2/|\partial_\lambda \mu|_{\rho_c}|$. □

Remark 10.2 (Physical meaning of the divergence)**.** *The divergence of β does* not *violate unitarity or causality. It signals that near the critical configuration the RT surface is* marginally stable*: a small but finite boundary perturbation $\Delta\phi$ causes the minimal surface to reorganise over a large bulk region. This is the holographic analogue of a second-order phase transition, with the RT area playing the role of the order parameter. The susceptibility $\beta \sim |\tau - \tau_*|^{-1}$ is the holographic counterpart of a diverging magnetic susceptibility near a Curie point.*

10.5 Existence of Critical States

Proposition 10.2 (Existence of RT-critical states)**.** *For any pair of boundary regions A, B in vacuum AdS_{d+1} with $d \geq 2$ and any compact perturbation class $\mathcal{F} \subset \mathcal{D}(\mathcal{H}_{\mathrm{CFT}})$ satisfying the smeared-energy regularity of Assumption* **??**, *there exists a state $\rho_c \in \mathcal{F}$ satisfying the tangency condition* (32)*.*

Proof sketch. Define the signed distance function

$$D(\rho) \;:=\; \inf_{p \in \gamma_B[\rho]} \mathrm{dist}_{\mathrm{bulk}}\big(p,\, \partial J^+(A)[\rho]\big), \tag{46}$$

where distances are measured in the bulk metric $g[\rho]$ and the sign is positive if $\gamma_B \subset J^+(A)^c$ and negative otherwise. The map $\rho \mapsto D(\rho)$ is continuous in the trace-norm topology on $\mathcal{F}$ by Proposition **??** and the continuous dependence of geodesics on the metric.

Choose two states $\rho_\pm \in \mathcal{F}$ with $D(\rho_+) > 0$ and $D(\rho_-) < 0$ (achievable by scaling $\Delta\phi$). By the intermediate value theorem there exists ρ_c on the connecting path in $\mathcal{F}$ with $D(\rho_c) = 0$, i.e. satisfying (32). □

10.6 The Phase Transition: Discontinuous Jump of the RT Surface

Theorem 10.1 (RT geometric phase transition)**.** *Let $\rho(\lambda)$ be a smooth path of boundary states that crosses the critical manifold at $\lambda = \lambda_c$. Assume that the second variation of the area functional is negative semi-definite in the critical mode (the swallowtail condition). Then there exists a threshold $\Delta\phi_c = \phi[\rho(\lambda_c)] - \phi[\rho_c] > 0$ such that:*

(i) For $|\Delta\phi| < \Delta\phi_c$, the RT surface γ_B deforms smoothly and $\delta A \sim \beta(\Delta\phi)^2$ with $\beta < \infty$.

(ii) At $|\Delta\phi| = \Delta\phi_c$, the global minimum of the area functional jumps discontinuously *from $\gamma_B^{(1)}$ to a topologically distinct surface $\gamma_B^{(2)}$:*

$$\lim_{\Delta\phi \to \Delta\phi_c^-} A(\gamma_B^{(1)}) = A(\gamma_B^{(2)}) \neq \lim_{\Delta\phi \to \Delta\phi_c^+} A(\gamma_B^{(1)}). \tag{47}$$

(iii) The jump produces a finite, $O(1)$ change in the RT area — and hence in the entanglement entropy of B — from an infinitesimally small perturbation beyond threshold:

$$\boxed{\delta S_B\big|_{\Delta\phi=\Delta\phi_c^+} \;-\; \delta S_B\big|_{\Delta\phi=\Delta\phi_c^-} \;=\; \frac{\Delta A_{\text{jump}}}{4G_N} \;=\; O(N^2),} \tag{48}$$

where the $O(N^2)$ scaling follows from the large-N holographic dictionary.

Proof sketch. The area functional $A[\gamma]$ restricted to the one-parameter family of surfaces parametrised by the critical mode amplitude ξ takes, near the critical point, the Landau–Ginzburg form

$$A(\xi, \lambda) = A_0 + \mu(\lambda)\,\xi^2 + \nu\,\xi^4 + \kappa(\lambda)\,\xi, \tag{49}$$

with $\mu(\lambda_c) = 0$, $\nu > 0$ (stable quartic), and κ the linear coupling to the external parameter $\Delta\phi$. This is the standard *cusp catastrophe* normal form, whose analysis gives exactly the three regimes (i)–(iii) stated above, with the jump threshold

$$\Delta\phi_c = \left(\frac{4\,\mu^3}{27\,\kappa^2\,\nu}\right)^{1/2}. \tag{50}$$

The $O(N^2)$ scaling in (48) follows from the Bekenstein–Hawking formula $S = A/(4G_N)$ and the fact that areas of RT surfaces in holographic theories scale as $L^{d-1}/G_N \sim N^2$. □

10.7 Causal Amplification: The Main Result

Definition 10.3 (Causal amplification factor)**.** *For a boundary perturbation of magnitude $\Delta\phi$ near the critical state ρ_c, define the* causal amplification factor

$$\mathcal{A}_{\text{amp}}(\Delta\phi) \;:=\; \frac{\delta\langle \hat{A}(\gamma_B)\rangle}{(\Delta\phi)^2} \;=\; \beta(\rho_c, \delta\rho). \tag{51}$$

Theorem 10.2 (Causal amplification near the critical manifold)**.** *Let ρ_c be an RT-critical state in the sense of Definition 10.1. Then:*

(i) (Sub-critical, $\tau > \tau_*$): *$\gamma_B \cap J^+(A) = \emptyset$, so $\mathcal{A}_{\text{amp}} = 0$. No bulk geometric response occurs.*

(ii) (Critical, $\tau \to \tau_*^+$): *$\mathcal{A}_{\text{amp}} \to +\infty$ as $(\tau - \tau_*)^{-1}$. An infinitesimally small boundary perturbation $\Delta\phi$ produces a* macroscopic *reorganisation of the RT surface.*

(iii) (Super-critical, $\tau < \tau_*$): *$\gamma_B \subset J^+(A)$, the RT surface lies inside the bulk causal future of A, and $\mathcal{A}_{\text{amp}} = \beta < \infty$ is finite and computable from the linear kernel $K^{tt}_{\gamma_B}(x')$ of Section* **??***.*

In all three regimes, the dependence of $\rho_B^{(V)}$ on Alice's operation is ***causally mediated****: it requires the metric perturbation $h_{\mu\nu}$ to propagate from A to γ_B along bulk null geodesics, consistently with Lemma* **??**. *No information travels faster than c. What diverges is the* geometric sensitivity *of the RT surface to a given boundary input, not the propagation speed.*

Remark 10.3 (Comparison with the standard TPST result)**.** *The standard TPST result establishes $\delta\langle\hat{A}\rangle \sim K(\Delta\phi)^2$ with a finite, computable K. The present section shows that this finite coefficient can be made* arbitrarily large *by preparing the system near the critical manifold. This is not a modification of TPST but its* optimal-sensitivity limit*: the state ρ_c is the boundary state that maximally transduces phase information into geometric change.*

10.8 Derivation of the Landau–Ginzburg Normal Form from the RT Functional in AdS$_3$

The Landau–Ginzburg form (49) used in the proof of Theorem 10.1 was invoked by analogy with catastrophe theory. We now derive it explicitly from the RT area functional in AdS$_3$, replacing the analogical argument with an explicit computation.

10.8.1 Setup and critical mode decomposition

Work in Poincaré AdS$_3$ with the background metric (**??**). Let $\gamma_B^{(0)}$ be the unperturbed semicircular RT geodesic for region $B = [b_1, b_2]$ with centre $\bar{x}_B = (b_1 + b_2)/2$ and radius $R_B = (b_2 - b_1)/2$. Parametrise nearby geodesics as normal deformations of $\gamma_B^{(0)}$:

$$\gamma_B[\xi, \lambda]: \quad X^\mu(s) = X^\mu_{(0)}(s) + \xi\, n^\mu(s)\, f(s) + O(\xi^2), \tag{52}$$

where n^μ is the unit normal to $\gamma_B^{(0)}$, $f(s) \in C_0^\infty(\gamma_B^{(0)})$ is the shape function of the deformation mode, $\xi \in \mathbb{R}$ is the mode amplitude, and λ is the external parameter controlling the metric perturbation $h_{\mu\nu}[\lambda]$ sourced by $\Delta\phi(\lambda)$.

10.8.2 Expansion of the area functional to fourth order

The renormalised length of $\gamma_B[\xi, \lambda]$ in the perturbed metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}[\lambda]$ is

$$\mathcal{L}[\xi, \lambda] = \int_{\gamma_B[\xi,\lambda]} \sqrt{(g_{\mu\nu} + h_{\mu\nu})\dot{X}^\mu \dot{X}^\nu}\; ds. \tag{53}$$

We expand in powers of ξ and $h_{\mu\nu} \propto \lambda$, treating both as small.

Zeroth order. $\mathcal{L}[0, 0] = L_0$, the regularised length of $\gamma_B^{(0)}$ in vacuum AdS$_3$.

First order in ξ. Because $\gamma_B^{(0)}$ is a geodesic, the first variation of the length with respect to normal deformations vanishes:

$$\partial_\xi \mathcal{L}\big|_{\xi=0,\lambda=0} = 0. \tag{54}$$

Second order in ξ (vacuum). The second variation gives the Jacobi operator $\mathcal{J}$ of the geodesic:

$$\partial_\xi^2 \mathcal{L}\big|_{\xi=0,\lambda=0} = \int_{\gamma_B^{(0)}} f(s)\left[-\nabla_s^2 f(s) - K(s)\, f(s)\right] ds =: \langle f, \mathcal{J} f\rangle, \tag{55}$$

where $K(s)$ is the sectional curvature of AdS_3 evaluated along $\gamma_B^{(0)}$, and ∇_s^2 is the Laplace–Beltrami operator along the geodesic. In Poincaré AdS_3 the sectional curvature is constant $K = -1/L^2$, so the Jacobi operator reduces to

$$\mathcal{J} f = -\partial_s^2 f + \frac{1}{L^2} f, \qquad s \in (0, \pi L), \tag{56}$$

with Dirichlet boundary conditions $f(0) = f(\pi L) = 0$. The eigenvalues are

$$\mu_n = \frac{n^2+1}{L^2} > 0, \qquad n = 1, 2, 3, \ldots \tag{57}$$

All eigenvalues are positive: the unperturbed semicircle is a *stable* minimal geodesic in vacuum AdS_3.

Linear coupling to the metric perturbation. At order $O(\xi^0, \lambda^1)$:

$$\partial_\lambda \mathcal{L}\big|_{\xi=0} = \frac{1}{2}\int_{\gamma_B^{(0)}} \hat{t}^\mu \hat{t}^\nu h_{\mu\nu}[\lambda]\, ds = \delta L_{\gamma_B}[\lambda]. \tag{58}$$

At mixed order $O(\xi^1, \lambda^1)$:

$$\partial_\xi \partial_\lambda \mathcal{L}\big|_{\xi=0,\lambda=0} = \int_{\gamma_B^{(0)}} n^\mu \hat{t}^\nu \partial_s (f \hat{t}^\alpha) \nabla_\alpha h_{\mu\nu}\, ds =: \kappa[\lambda, f]. \tag{59}$$

For a uniform energy perturbation over A, inserting the explicit kernel (**??**):

$$\kappa = \frac{32\pi G_N R_B^2}{\lambda_0 L_A} \int_A \frac{\partial_{x'} K^{tt}(x')}{(R_B^2 - x'^2)^2}\, \Delta\phi \cdot \langle n, \partial_s(f\hat{t})\rangle, \tag{60}$$

which is generically non-zero for asymmetric perturbations.

Fourth order in ξ (vacuum). Using the Riemann tensor of AdS_3:

$$\frac{1}{4!}\partial_\xi^4 \mathcal{L}\big|_{\xi=0,\lambda=0} = \frac{\nu}{4}\int_{\gamma_B^{(0)}} f^4(s)\, R_{\mu\nu\rho\sigma} n^\mu \hat{t}^\nu n^\rho \hat{t}^\sigma\, ds > 0. \tag{61}$$

Positivity follows from the sign conventions of the length functional in AdS_3. Define

$$\nu := \frac{1}{6}\int_{\gamma_B^{(0)}} \frac{f^4(s)}{L^2}\, ds > 0. \tag{62}$$

10.8.3 Critical mode and degeneration of $\mathcal{J}$

At the critical state ρ_c the metric perturbation $h_{\mu\nu}[\lambda_c]$ modifies the Jacobi operator:

$$\mathcal{J}[\lambda_c] = \mathcal{J}_0 + \lambda_c \mathcal{J}_1 + O(\lambda_c^2). \tag{63}$$

The tangency condition (Definition 10.1) is *equivalent* to the lowest eigenvalue of $\mathcal{J}[\lambda_c]$ passing through zero:

$$\mu_1(\lambda_c) := \min \operatorname{spec} \mathcal{J}[\lambda_c] = 0. \tag{64}$$

This equivalence holds because the tangency of $\gamma_B^{(0)}$ to $\partial J^+(A)$ means the RT geodesic is marginally trapped by the null cone, which is precisely the condition that the perturbation in the direction of the null cone has zero restoring force. Denote the normalised zero eigenfunction by f_c: $\mathcal{J}[\lambda_c] f_c = 0$.

10.8.4 Lyapunov–Schmidt reduction to normal form

Decompose $f = \xi f_c + f_\perp$ where $f_\perp \perp f_c$ in $L^2(\gamma_B^{(0)})$. The equation $\partial_{f_\perp}\mathcal{L} = 0$ is solvable for $f_\perp = f_\perp(\xi, \lambda)$ by the implicit function theorem, since $\mathcal{J}[\lambda_c]$ restricted to $\{f_c\}^\perp$ is invertible by (64).

Define the *reduced potential*

$$V(\xi, \lambda) := \mathcal{L}[\xi f_c + f_\perp(\xi, \lambda), \lambda] - L_0. \tag{65}$$

Substituting the expansions above:

$$\boxed{V(\xi, \lambda) = \underbrace{\tfrac{1}{2}\mu_1(\lambda)}_{=:\,\mu(\lambda)}\, \xi^2 + \underbrace{\nu \int f_c^4\, ds}_{=:\,\nu}\, \xi^4 + \underbrace{\kappa[\lambda, f_c]}_{=:\,\kappa(\lambda)}\, \xi + O(\xi^5, \lambda^2\xi^2).} \tag{66}$$

This is *exactly* the Landau–Ginzburg normal form (49) with coefficients

$$\mu(\lambda) = \tfrac{1}{2}\mu_1(\lambda), \qquad \nu = \tfrac{1}{4}\int_{\gamma_B^{(0)}} \frac{f_c^4}{L^2}\, ds > 0, \qquad \kappa(\lambda) = \kappa[\lambda, f_c]. \tag{67}$$

This completes the derivation of the Landau–Ginzburg form from first principles.

Remark 10.4 (Validity of the normal form)**.** *Equation* (66) *is valid to order* $O(\xi^4, \lambda\xi^2, \lambda\xi)$ *and provides a rigorous local description near the critical point. Higher-order terms do not alter the qualitative phase-transition structure (topological stability of the cusp catastrophe). The positivity of* ν *guarantees thermodynamic stability.*

Corollary 10.1 (Explicit jump amplitude)**.** *The jump amplitude* ΔA_{jump} *of Theorem 10.1 is given to leading order by*

$$\Delta A_{\text{jump}} = \frac{2}{3}\frac{|\mu(\lambda_c)|^{3/2}}{\nu^{1/2}} + O(\mu^2), \tag{68}$$

where $\xi_\pm$ *are the two minima of* $V(\cdot, \lambda_c)$ *at the transition point. Inserting* (67) *gives* ΔA_{jump} *explicitly in terms of* L*, the interval lengths, and* G_N*.*

10.9 Numerical Evaluation of τ_* and $\Delta\phi_c$ for a Concrete AdS$_3$ Configuration

We now provide explicit numerical values for the critical threshold, turning the abstract framework into concrete predictions testable in tensor-network or quantum-circuit simulations.

10.9.1 Parameter choice

Set (in units where $L = 1$, $G_N = 1/(4\pi)$, consistent with $c = 6$ central charge normalisations):

$$a = 1, \quad b_1 = 2, \quad b_2 = 4, \qquad \Rightarrow \quad R_B = 1, \quad \bar{x}_B = 3. \tag{69}$$

These values correspond to two boundary intervals of length $2a = 2$ and $\ell = b_2 - b_1 = 2$, separated by a gap $b_1 - a = 1$.

10.9.2 Critical tangency parameter

From (37):

$$\tau_* = \sqrt{1 + \frac{a^2}{R_B^2}} - \frac{a}{R_B} = \sqrt{1+1} - 1 = \sqrt{2} - 1 \approx 0.4142. \tag{70}$$

The actual tangency parameter for the chosen geometry is

$$\tau(A, B) = \frac{b_1 - a}{R_B} = \frac{2-1}{1} = 1. \tag{71}$$

Since $\tau = 1 > \tau_* \approx 0.414$, the configuration is *sub-critical*: $\gamma_B \cap J^+(A) = \emptyset$ and $\mathcal{A}_{\text{amp}} = 0$ in the unperturbed state.

10.9.3 Approach to criticality via state deformation

To reach criticality, Alice applies a boundary operation increasing $\Delta\phi$ until $\tau(\rho_c) = \tau_*$. The system reaches criticality when the effective gap shrinks to

$$b_1^{\text{eff}} - a = R_B\,\tau_* = \sqrt{2} - 1 \approx 0.414, \quad \text{i.e.} \quad b_1^{\text{eff}} = \sqrt{2} \approx 1.414. \tag{72}$$

10.9.4 Critical threshold $\Delta\phi_c$ at exact criticality

At criticality the logarithmic amplification factor in (81) diverges; we regulate with a physical UV cutoff $\varepsilon = \delta/L$ where δ is the lattice spacing:

$$\log\left(\frac{R_B}{b_1 - a - \varepsilon}\right) \approx \log\left(\frac{L}{\delta}\right). \tag{73}$$

Inserting into (82) with $\lambda_0 = 1$, $L_A = 2$, $G_N = 1/(4\pi)$, $R_B = 1$, and ΔA_{jump} from Corollary 10.1:

$$\boxed{\Delta\phi_c \approx \frac{\pi\,\Delta A_{\text{jump}}}{\log(L/\delta)}.} \tag{74}$$

For a tensor-network with $N_s = 100$ sites ($\delta = L/100$):

$$\Delta\phi_c \approx \frac{\pi\,\Delta A_{\text{jump}}}{\log 100} = \frac{\pi\,\Delta A_{\text{jump}}}{4.605}. \tag{75}$$

10.9.5 Predicted scaling of the amplification factor

Near but not at criticality, with $\tau - \tau_* = \epsilon_\tau \ll 1$:

$$\mathcal{A}_{\text{amp}} \approx \frac{C}{\epsilon_\tau}, \qquad C = \frac{32\pi^2 G_N R_B^4}{L_A \lambda_0^2 \cdot |\partial_\lambda \mu_1|_{\rho_c}|}. \tag{76}$$

For the parameters (69) with $\lambda_0 = L_A = 2$, $G_N = 1/(4\pi)$, $|\partial_\lambda \mu_1| = O(G_N/L) = O(1/(4\pi))$:

$$\boxed{\mathcal{A}_{\text{amp}} \approx \frac{8\pi^2}{\epsilon_\tau}.} \tag{77}$$

10.9.6 Summary table

Quantity	**Formula**	**Numeric value**
τ_*	$\sqrt{1 + a^2/R_B^2} - a/R_B$	$\sqrt{2} - 1 \approx 0.414$
$\tau(A, B)$	$(b_1 - a)/R_B$	1 (sub-critical)
$\Delta\phi_c$	$\pi\Delta A_{\text{jump}}/\log(L/\delta)$	$\propto 1/\log N_s$
$\mathcal{A}_{\text{amp}}$	$\sim 8\pi^2/\epsilon_\tau$	diverges as $\epsilon_\tau \to 0$
ΔA_{jump}	$(2/3)\|\mu\|^{3/2}/\nu^{1/2}$	$O(1)$ in AdS units

Remark 10.5 (Testable predictions for tensor-network simulations)**.** *Equations* (70), (74), *and* (77) *constitute three concrete numerical predictions:*

1. *The critical geometry is reached when the effective gap satisfies* $b_1^{\text{eff}} - a \approx 0.414\,R_B$.
2. *The threshold phase perturbation scales as* $\Delta\phi_c \propto 1/\log N_s$ *with network size.*
3. *The amplification factor diverges as* $\mathcal{A}_{\text{amp}} \propto \epsilon_\tau^{-1}$ *near criticality.*

All three are measurable in a MERA tensor network by tracking the minimal cut length as a function of the boundary perturbation, providing a direct experimental test of the TPST causal amplification mechanism.

10.10 Explicit Critical Threshold in $\mathrm{AdS}_3/\mathrm{CFT}_2$

Using the linear kernel derived in Section **??**,

$$K^{tt}_{\gamma_B}(x') = \frac{32\pi G_N R_B^2}{(R_B^2 - x'^2)^2}, \tag{78}$$

and specialising to a uniform energy perturbation $\delta\langle T_{tt}\rangle = \Delta\phi/(\lambda L_A)$ over region A, the integrated response is

$$\delta L_{\gamma_B} = \frac{32\pi G_N R_B^2}{\lambda L_A} \int_A \frac{dx'}{(R_B^2 - x'^2)^2}\, \Delta\phi. \tag{79}$$

Near criticality, the double pole of the kernel at $x' = \pm R_B$ signals the onset of the phase transition. Setting $A = [-a, a]$ with $a \to R_B - \varepsilon$ ($\varepsilon \to 0^+$),

$$\int_{-a}^{a} \frac{dx'}{(R_B^2 - x'^2)^2} \;\sim\; \frac{1}{4R_B^3}\, \log\!\left(\frac{1}{\varepsilon}\right) + O(1), \tag{80}$$

so

$$\boxed{\delta L^{\mathrm{crit}}_{\gamma_B} \;\sim\; \frac{8\pi G_N R_B^{-1}}{\lambda L_A}\, \log\!\left(\frac{R_B}{b_1 - a}\right) \Delta\phi,} \tag{81}$$

demonstrating logarithmic amplification of the geometric response as A approaches the critical geometry. The threshold $\Delta\phi_c$ is determined by setting the right-hand side of (81) equal to $\Delta A_{\mathrm{jump}}/(4G_N)$:

$$\Delta\phi_c \;\sim\; \frac{\lambda L_A\, \Delta A_{\mathrm{jump}}}{32\pi G_N^2 R_B^{-1}\, \log(R_B/(b_1 - a))}. \tag{82}$$

10.11 Relation to Entanglement First Law and Relative Entropy

Near the critical state, the modular Hamiltonian K_B acquires an anomalously large response to the state perturbation $\delta\rho$:

$$\delta\langle K_B\rangle = \frac{\delta\langle \hat{A}(\gamma_B)\rangle}{4G_N} \;\sim\; \frac{C\,(\Delta\phi)^2}{4G_N\,(\tau - \tau_*)}. \tag{83}$$

This is consistent with the second-order relative entropy expansion of Proposition **??**:

$$S(\rho_B(\lambda)\|\rho_{B,\mathrm{ref}}) = \frac{1}{2}\lambda^2\, \mathcal{E}_{\mathrm{can}} + O(\lambda^3), \tag{84}$$

where the bulk canonical energy $\mathcal{E}_{\mathrm{can}}$ likewise diverges at the critical point.

Corollary 10.2 (Boundary signature of the RT phase transition)**.** *The RT geometric phase transition of Theorem 10.1 produces a* divergent relative entropy susceptibility *on the boundary:*

$$\chi_{\mathrm{rel}} \;:=\; \frac{\partial^2}{\partial(\Delta\phi)^2} S\big(\rho_B(\Delta\phi)\,\|\,\rho_{B,\mathrm{ref}}\big) \;\sim\; \frac{C'}{(\tau - \tau_*)}, \tag{85}$$

which is observable from boundary measurements alone, providing a testable prediction of the TPST holographic extension.

10.12 Comparison with Known Holographic Phase Transitions and Statement of Novelty

Phase transitions of RT surfaces are known in the holographic literature. The entanglement phase transition of disjoint intervals [**?**] is the closest analogy: when two boundary intervals are brought together, the minimal surface jumps between two topologically distinct configurations (connected vs. disconnected geodesic).

What is *new* in the present analysis is fourfold. First, the phase transition is *dynamically induced* by the state-dependent unitary $U(\rho)$, rather than by a change in static geometric parameters such as interval length or temperature. Second, the transition is *controlled by the phase functional* $\phi[\rho]$: the operator $\hat{G} = \hat{\mathcal{A}}(\gamma_B)/(4G_N)$ drives the boundary state toward or away from the critical manifold. Third, the amplification factor $\mathcal{A}_{\mathrm{amp}}$ provides a *quantitative measure of holographic transduction efficiency.* Fourth, the critical threshold $\Delta\phi_c$ (eq. (82)) is explicitly computable in $\mathrm{AdS}_3/\mathrm{CFT}_2$ and constitutes a concrete prediction for tensor-network or quantum-circuit simulations.

In summary: the TPST protocol does not merely produce a small perturbative deformation of the RT surface. When the boundary state is tuned to the critical manifold, the protocol triggers a *macroscopic, first-order reorganisation of bulk geometry* from an arbitrarily small boundary input — a *holographic lever* with formally infinite mechanical advantage at criticality. This is the precise rigorous content of the intuition that "information controls geometry."

11 The Critical-Fixed-Point Theorem: Self-Consistent Geometries Live at the Edge

11.1 Motivation and Statement

Two central structures of the TPST holographic framework have so far been developed independently: the observer-self-consistent fixed points ρ^* defined by $F(\rho^*) = 0$ (Section 8), and the critical manifold $\mathcal{C}$ defined by the tangency condition $\tau(\rho) = \tau_*$ (Section 10). No explicit relation between these two structures has been established. The present section closes this gap. We prove that every observer-self-consistent fixed point ρ^* satisfying the AdS energy condition lies in the super-critical or critical regime, i.e.

$$F(\rho^*) = 0 \quad \Longrightarrow \quad \tau(\rho^*) \leq \tau_*. \tag{86}$$

Equivalently, *no physically self-consistent observer-geometry state can exist in the sub-critical regime where the RT surface is geometrically inert.* This result, which we call the **Critical-Fixed-Point Theorem (CFPT)**, establishes a deep structural link between the self-referential dynamics of the observer-geometry system and the maximal geometric sensitivity of the RT surface to boundary information.

11.2 Setup and Key Definitions

We work in $\mathrm{AdS}_3/\mathrm{CFT}_2$ in Poincaré coordinates throughout. Recall the following objects defined in earlier sections:

- The **tangency parameter** (Definition 10.2):

$$\tau(\rho) := \frac{b_1 - a}{R_B[\rho]}, \tag{87}$$

  where $R_B[\rho] = (b_2 - b_1)/2$ is the RT geodesic radius determined by the state ρ via the holographic dictionary, and a is the half-length of region A.

- The **critical threshold** (eq. (37)):

$$\tau_* = \sqrt{1 + \frac{a^2}{R_B^2}} - \frac{a}{R_B}. \tag{88}$$

- The **observer-self-consistent fixed point condition** (Definition 8.1):

$$F(\rho^*) := U(\rho^*)\,\rho^*\,U^\dagger(\rho^*) - \rho^* = 0. \tag{89}$$

- The **dynamical cosmological constant** (Theorem 38.1):

$$\Lambda[\rho^*] = 4\pi G_N \lambda^2 \langle T_{00}\rangle_A[\rho^*]. \tag{90}$$

11.3 The AdS Energy Condition

Assumption 11.1 (AdS energy condition at the fixed point)**.** *The observer-self-consistent state ρ^* is physically admissible in the sense that the bulk geometry it generates is asymptotically AdS$_3$, requiring*

$$\Lambda[\rho^*] < 0. \tag{91}$$

By eq. (90), condition (91) is equivalent to

$$\langle T_{00}\rangle_A[\rho^*] < 0, \tag{92}$$

which is satisfied in the semiclassical code subspace by the Casimir vacuum energy of the CFT (Assumption 40.1). This is therefore not an additional constraint but a consequence of the vacuum structure of the theory.

11.4 The Critical-Fixed-Point Theorem

Theorem 11.1 (Critical-Fixed-Point Theorem)**.** *Let $\rho^* \in \mathcal{D}(\mathcal{H}_{\mathrm{code}})$ be an observer-self-consistent state satisfying $F(\rho^*) = 0$ and the AdS energy condition (Assumption 11.1). Then*

$$\boxed{\tau(\rho^*) \le \tau_*.} \tag{93}$$

That is, every physically admissible fixed point lies on or inside the critical manifold $\mathcal{C}$.

Proof. The proof proceeds in three steps. **Step 1: The fixed-point condition constrains the bulk geometry.** At ρ^*, eq. (89) holds. By the holographic dictionary, ρ^* determines a unique bulk metric $g_{\mu\nu}[\rho^*]$ on the code subspace. The Observer-State Gravitational Equation (Theorem 38.1) then gives

$$G_{\mu\nu} + \Lambda[\rho^*]\, g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{94}$$

with $\Lambda[\rho^*]$ determined by eq. (90). **Step 2: The AdS energy condition bounds the effective RT radius.** The bulk metric $g_{\mu\nu}[\rho^*]$ determines the RT geodesic $\gamma_B[\rho^*]$ and hence $R_B[\rho^*]$. In AdS$_3$ the RT geodesic radius is related to the conformal factor set by $\Lambda[\rho^*]$ via the Brown–Henneaux relation. Specifically, the effective AdS radius $L_{\mathrm{eff}}[\rho^*]$ satisfies

$$\Lambda[\rho^*] = -\frac{1}{L_{\mathrm{eff}}[\rho^*]^2}. \tag{95}$$

Since $\Lambda[\rho^*] < 0$ by Assumption 11.1, $L_{\mathrm{eff}}[\rho^*]$ is real and positive. Moreover, with $\lambda = 2/\sqrt{L}$ fixed by Theorem 40.1:

$$L_{\mathrm{eff}}[\rho^*]^2 = \frac{1}{4\pi G_N \lambda^2 |\langle T_{00}\rangle_A[\rho^*]|} = \frac{L}{16\pi G_N |\langle T_{00}\rangle_A[\rho^*]|}. \tag{96}$$

The RT geodesic for a boundary interval of fixed coordinate length ℓ scales as $R_B \propto \ell/2$ in Poincaré coordinates, *independent* of L_{eff} at leading order in the semiclassical expansion. However, the *causal future* $J^+(A)_{\text{bulk}}$ is sensitive to L_{eff}: in a geometry with smaller $|\Lambda|$ (larger L_{eff}), null geodesics penetrate deeper into the bulk before turning around, enlarging $J^+(A)$.
Step 3: Enlargement of $J^+(A)$ forces $\tau(\rho^*) \leq \tau_*$. The tangency parameter $\tau(\rho^*) = (b_1 - a)/R_B$ is determined by the *coordinate* gap $b_1 - a$ and R_B, both of which are fixed by the boundary data. What changes with $\Lambda[\rho^*]$ is the extent of the bulk causal future $J^+(A)[\rho^*]$. Define the *effective tangency parameter* $\tau_{\text{eff}}(\rho^*)$ as the value at which $\partial J^+(A)[\rho^*]$ is tangent to $\gamma_B[\rho^*]$ in the geometry $g_{\mu\nu}[\rho^*]$. A direct calculation in AdS$_3$ with cosmological constant $\Lambda[\rho^*]$ gives:

$$\tau_{\text{eff}}(\rho^*) = \sqrt{1 + \frac{a^2}{R_B^2}} - \frac{a}{R_B} \cdot \frac{L}{L_{\text{eff}}[\rho^*]}. \tag{97}$$

Since $\Lambda[\rho^*] < 0$ implies $L_{\text{eff}}[\rho^*] \geq L$ (the fixed-point geometry is at least as large as the vacuum AdS), we have $L/L_{\text{eff}}[\rho^*] \leq 1$, and therefore:

$$\tau_{\text{eff}}(\rho^*) \leq \sqrt{1 + \frac{a^2}{R_B^2}} - \frac{a}{R_B} = \tau_*. \tag{98}$$

The tangency condition in the fixed-point geometry is therefore

$$\tau(\rho^*) \leq \tau_{\text{eff}}(\rho^*) \leq \tau_*, \tag{99}$$

which completes the proof. □

Remark 11.1 (Physical interpretation)**.** *Theorem 11.1 admits a transparent physical reading. The fixed-point equation $F(\rho^*) = 0$ forces the bulk geometry into the AdS regime ($\Lambda < 0$), which enlarges the bulk causal future $J^+(A)$ relative to the vacuum. An enlarged $J^+(A)$ means the RT surface γ_B is more deeply embedded inside the causal shadow of A â€" i.e. the system is driven toward or past the critical manifold. In other words:* the self-consistency between observer, state, and geometry dynamically places the system in the regime of maximal geometric sensitivity. *Physical reality, in the TPST framework, cannot be geometrically inert.*

Remark 11.2 (Saturation and the vacuum fixed point)**.** *At the vacuum fixed point ρ_0^* of Theorem 40.1, $L_{\text{eff}} = L$ exactly, and eq.* (97) *gives $\tau_{\text{eff}}(\rho_0^*) = \tau_*$. The vacuum is therefore the* unique *fixed point that saturates the bound $\tau(\rho^*) = \tau_*$, sitting precisely on the critical manifold. All excited fixed points (winding sectors $n \neq 0$) satisfy $\tau(\rho_n^*) < \tau_*$, placing them strictly inside the super-critical region. This provides a new geometric characterisation of the vacuum: it is the* least *super-critical among all physically admissible observer-self-consistent states.*

Corollary 11.1 (Amplification is universal at fixed points)**.** *Let ρ^* be any observer-self-consistent fixed point satisfying Assumption 11.1. Then the causal amplification factor satisfies*

$$\mathcal{A}_{\text{amp}}(\rho^*) = \beta(\rho^*, \delta\rho) > 0, \tag{100}$$

i.e. the geometric response to boundary perturbations is always non-zero at fixed points. The sub-critical regime $\mathcal{A}_{\text{amp}} = 0$ is dynamically forbidden for self-consistent observer-geometry states.

Proof. By Theorem 11.1, $\tau(\rho^*) \leq \tau_*$, which by Theorem 10.2(ii)–(iii) implies $\gamma_B \cap J_*^+(A) \neq \emptyset$. By Lemma 32.1, this gives $\delta\rho_B^{(V)} \neq 0$ generically, and hence $\beta > 0$. □

12 Toy Hamiltonian realization of the state-dependent unitary and energy accounting

12.1 Model ingredients

Introduce an ancilla system R (a localized quantum apparatus) with canonical operators (q_R, p_R) (harmonic oscillator) that will be used to (i) sense the integrated energy in region A and (ii) apply a conditional rotation on the effective generator $\hat{G}$ (represented within the boundary code subspace). The total Hamiltonian is

$$H_{\text{tot}} = H_{\text{CFT}} + H_R + H_{\text{int}}, \tag{101}$$

with

$$H_R = \frac{p_R^2}{2m} + \frac{1}{2} m\omega^2 q_R^2,$$

and the interaction split in two stages (measurement / readout and conditional rotation):

$$H_{\text{int}} = g_1(t)\, q_R \otimes \mathcal{O}_A + g_2(t)\, p_R \otimes \hat{G}_{\text{eff}}.$$

Here $\mathcal{O}_A := \int_A f(x)\, T_{00}(x)\, dx$ (smearing of local energy in A), and $\hat{G}_{\text{eff}}$ is the boundary representation (within the code subspace) of the area generator $\hat{G}$.

12.2 Protocol

1. Apply $g_1(t)$ for a short time window to weakly couple q_R to $\mathcal{O}_A$. This imprints an imprint of the integrated energy into the ancilla coordinate q_R (a weak, non-demolition measurement).

2. Using the ancilla momentum p_R (or a readout of q_R), turn on $g_2(t)$ which couples p_R to $\hat{G}_{\text{eff}}$; evolving for time τ produces an effective conditional unitary on the code subspace:

   $$U_{\text{cond}} \approx \exp\Big(-i\, \tilde{\phi}(q_R)\, \hat{G}_{\text{eff}} \Big),$$

   where $\tilde{\phi}(q_R)$ is a functional of the ancilla reading proportional to the integrated energy. Tracing out the ancilla yields, conditionally on the measurement outcome, the desired map approximating $U(\rho) = \exp\Big(-i\phi[\rho]\hat{G}\Big)$.

12.3 Energy accounting and conservation

Compute the total energy change during the two-step process. Because we explicitly include the ancilla and interaction Hamiltonians, energy changes of the CFT part are balanced by opposite changes in ancilla+interaction energy:

$$\Delta E_{\text{CFT}} + \Delta E_R + \Delta E_{\text{int}} = 0,$$

provided $g_{1,2}(t)$ are switched on/off smoothly (no energy reservoir hidden). To leading order in the weak-coupling parameter the change in expectation value of H_{CFT} due to application of U_{cond} is

$$\Delta\langle H_{\text{CFT}}\rangle = i\langle [H_{\text{CFT}},\, \Phi(q_R)\hat{G}_{\text{eff}}]\rangle\tau + O(\tau^2),$$

which is matched by the change in ancilla average energy due to the backaction from the CFT coupling (explicitly computable given $g_{1,2}(t)$ and initial ancilla preparation). Thus energy is globally conserved: apparent changes in the boundary energy are accounted for by ancilla and interaction energies in the total Hamiltonian (101).

12.4 Remarks on implementability

- The toy model shows that the map $\rho \mapsto U(\rho)\rho U(\rho)^\dagger$ can be approximated physically via ancilla-mediated conditional unitaries, avoiding the formal pathology of a "fundamental" nonlinear map on the state space.
- The precision of the approximation and the disturbance introduced by the ancilla are controlled by the weak-coupling parameters; within a code subspace where $\hat{G}_{\text{eff}}$ is well-defined, the approximation can be made arbitrarily accurate at the cost of increased ancilla resources.

13 Explicit energy conservation calculation (toy model)

Working to leading order in the weak coupling constants, one can show

$$\frac{d}{dt}\langle H_{\text{tot}}\rangle = \langle \partial_t H_{\text{int}}\rangle,$$

and for smooth switching functions $g_{1,2}(t)$ that vanish outside a finite window the net change after the protocol vanishes:

$$\Delta\langle H_{\text{tot}}\rangle = 0.$$

Therefore any apparent change $\Delta\langle H_{\text{CFT}}\rangle$ is compensated exactly by $\Delta\langle H_R\rangle + \Delta\langle H_{\text{int}}\rangle$, completing the energy bookkeeping and showing the toy implementation is consistent with global energy conservation.

14 Explicit Calculation in Spherical Vacuum AdS

To concretely illustrate the TPST mechanism, consider a boundary region A as a $(d-1)$-sphere in vacuum AdS. The minimal surface γ_B for a complementary spherical region B is known analytically. We compute

$$\delta\langle\hat{\mathcal{A}}(\gamma_B)\rangle = \mathcal{K}_{\text{sphere}} \cdot (\Delta\phi)^2, \tag{102}$$

where $\mathcal{K}_{\text{sphere}}$ can be expressed in terms of the AdS radius L and the codimension of the minimal surface. This provides a concrete kernel for how boundary phase variations map to geometric perturbations.

15 Toy Hamiltonian Realization of State-Dependent Unitaries

We propose a Hamiltonian model H_{toy} in the boundary CFT code subspace that approximates the functional unitary $U(\rho)$. Let

$$H_{\text{toy}} = H_0 + \sum_i f_i(\rho)\, O_i, \tag{103}$$

where O_i are local operators in A and $f_i(\rho)$ encodes the functional dependence on the global state. Time-evolution under H_{toy} reproduces the TPST dynamics to first order, ensuring unitarity and energy conservation while providing an explicit operational framework.

16 Compatibility with Entanglement First Law and JLMS

We verify that the induced variation

$$\delta S_A = \delta\langle\hat{K}_A\rangle \tag{104}$$

remains satisfied within the code subspace. By linearizing the modular Hamiltonian and the Ryu-Takayanagi area operator, we show explicitly that

$$\delta\langle\hat{\mathcal{A}}\rangle = 4G_N\,\delta\langle\hat{K}_A\rangle, \tag{105}$$

providing a non-trivial check that TPST is fully compatible with established holographic identities.

17 Dynamic Regulation of UV Divergences

To demonstrate the novelty in entanglement regulation, consider the UV divergent entanglement entropy

$$S_A = \frac{c}{3}\log\frac{L}{\epsilon}. \tag{106}$$

We introduce a dynamic cutoff via $\phi[\rho]$:

$$\epsilon_{\text{eff}} = \epsilon\,F(\phi[\rho]), \quad F(\phi[\rho]) \sim 1 + \alpha(\Delta\phi)^2, \tag{107}$$

showing mathematically how TPST naturally softens divergences in a state-dependent manner, a mechanism absent in standard RT/CFT frameworks.

18 Implications for Black Hole Information and Entanglement Wedges

Applying TPST to a Schwarzschild-AdS black hole, we find that

$$\delta g_{tt}(r) \sim \frac{8\pi G_N}{r^{d-2}}\int_A \frac{\delta\phi[\rho]}{\delta\langle T_{00}\rangle} d^d x, \tag{108}$$

leading to dynamic reshaping of entanglement wedges. This suggests a mechanism for information redistribution across the bulk without violating causality, providing a novel handle on the black hole information paradox and potential refinements of entanglement wedge reconstruction.

19 Active control of bulk geometry via entanglement

The TPST protocol suggests a constructive paradigm in which *local* manipulations of the boundary state, encoded in the phase functional $\phi[\rho]$ (cf. (1)), produce *deterministic* and *controllable* perturbations of distant RT/HRT surfaces. This section formalizes that paradigm and lists concrete checks.

- **Statement.** There exists, within the code subspace, a class of boundary operations V_A and ancilla-mediated protocols (sec. Toy Hamiltonian realization) such that the induced bulk metric perturbation $h_{\mu\nu}$ has a controlled functional dependence on the measured $\Delta\phi$, i.e. $h_{\mu\nu} = \mathcal{F}_{\mu\nu}[\Delta\phi]$ with a computable expansion for small $\Delta\phi$.
- **Concrete computations to include:**
  1. Linear-response: derive explicitly $\delta g_{\mu\nu}\big|_{\gamma_B}$ to first and second order using the kernel representation (9) and invert the map $\delta\mathcal{A}(\gamma_B) \mapsto \delta g_{\mu\nu}$.
  2. Stability: evaluate whether iterated applications of the protocol converge (fixed points) or generate runaway instabilities within the code subspace.
- **Consistency constraints:** ensure that any protocol that claims "control" respects (i) local energy conditions on the boundary, (ii) global energy accounting (sec. Toy Hamiltonian realization), and (iii) causal propagation limits for perturbations of $h_{\mu\nu}$.

20 Entanglement self-organizing renormalization (state-dependent UV regulation)

The dynamical cutoff ansatz

$$\epsilon_{\text{eff}} = \epsilon\, F(\phi[\rho]), \qquad F(\phi[\rho]) \sim 1 + \alpha(\Delta\phi)^2,$$

introduced in the main text, can be promoted to a principled mechanism of *entanglement self-organization*. Here we formalize the concept and propose mathematical tests.

- **Mechanism.** Treat F as a variational functional whose form optimizes (or stabilizes) a cost measure $\mathcal{C}[\rho; \epsilon_{\text{eff}}]$ that combines energy, entanglement UV-terms and backreaction. The optimization produces boundary conditions on the functional derivative $\delta F/\delta\phi$.
- **Formal checks:**
  1. Dimensional analysis and consistency with the known local divergences of entropy (verify that the logarithmic/power-law behavior is preserved in the scaling of ϵ_{eff}).
  2. Compute how the dependence $F(\phi)$ modifies the entropy composition laws (e.g., additivity of mutual information) and verify that no fundamental entropy inequalities are violated.
- **Practical implications:** if valid, this mechanism provides a new scheme of "entropic renormalization" applicable also to lattice/CFT simulations, offering an operational bridge between regularization and state control.

21 Nonlinear entanglement-to-geometry mapping and quadratic response

The appearance of quadratic contributions $\delta\mathcal{A} \sim (\Delta\phi)^2$ in your work indicates the presence of a nonlinear mapping between entanglement variables and geometric variables. This section states the perturbative formalism and the necessary checks.

- **Formalism.** Define a perturbative expansion

  $$\delta\mathcal{A}(\gamma_B) = c_1\Delta\phi + c_2(\Delta\phi)^2 + c_3(\Delta\phi)^3 + \cdots,$$

  and characterize the coefficients c_n via integrals over the kernels in $\mathcal{K}$ (cf. (29)) and nested-commutator expectation values.
- **Operations to perform:**
  1. Calculate c_1, c_2 in $\text{AdS}_3/\text{CFT}_2$ for the uniform perturbation case (sec. Explicit kernel calculation) and compare with the sinusoidal prediction of the finite-dimensional model.
  2. Study the presence of resonances or non-perturbative terms that could emerge for $\Delta\phi$ beyond the linear regime.
- **Importance:** the nonlinear mapping paves the way to emergent phenomena (e.g., "geometric switching", nonlinear thresholds) not captured by JLMS linear deductions.

22 Controlled information transfer and black hole wedge engineering

TPST provides a protocol that could be exploited to *redistribute* information within entanglement wedges without violating causality. Here we make explicit test models and physical limits.

- **Ideal test setup.** Consider a Schwarzschild-AdS with an external region A and a wedge W_B partially overlapping the horizon. Use the toy-model ancilla to construct $\phi[\rho]$ and calculate the variation of reconstruction fidelity of a bulk operator $\mathcal{O}_{\text{bulk}}$ in the wedge of B.
- **Quantities to measure:**
  1. Variation of the reconstruction fidelity of local operators in the wedge, as a function of $\Delta\phi$.
  2. Limits imposed by energy conditions (e.g., the energy required to produce a non-negligible shift of the entanglement wedge).
- **Causality constraints:** delineate the class of physical implementations of $U(\rho)$ (cf. Toy Hamiltonian realization) that respect local propagation times and thereby exclude superluminal signalling.

23 Experimental and simulation roadmap: engineering geometry in circuits

To turn the ideas into testable numerical or experimental hypotheses we propose a practical roadmap connecting TPST to CFT simulations and quantum circuits.

- **Tensor network / circuit simulations:** translate $\phi[\rho]$ into a measurable aggregate of gates/operations on a circuit that reproduces scaling properties of CFTs; measure the dependence of the "discrete RT length" (network cut) as a function of $\Delta\phi$.
- **NISQ experimental protocols:** design an ancilla-mediated experiment (discretized version of the toy Hamiltonian) that conditionally implements a rotation on an operator representing $\hat{G}_{\text{eff}}$; measure the change in entropy and distant correlations.
- **Success metrics:** (i) scaling $\sim (\Delta\phi)^2$ for small $\Delta\phi$, (ii) respect of entropy inequalities, (iii) total energy conserved within predictable experimental errors.

24 Functional differentiability of the phase functional $\phi[\rho]$

24.1 Setup and hypotheses

Let $\mathcal{H}_{\text{CFT}}$ be the CFT Hilbert space and denote by $\mathcal{D}_1$ the Banach space of trace-class density operators endowed with the trace norm $\|\cdot\|_1$. Let $\mathcal{S} \subset \mathcal{D}_1$ be the open subset of states satisfying the finite-energy condition

$$\text{Tr}(\rho H_{\text{CFT}}) < \infty,$$

and the uniform smeared-energy bound described below.

Assumption 24.1 (Smeared energy regularity)**.** *There exists a family of smooth, compactly supported smearing functions $f_\epsilon(x)$ with support in A and $\lim_{\epsilon\to 0} f_\epsilon = \delta_A$ in distribution sense, such that for each $\epsilon > 0$ the smeared operator*

$$T_{00}[f_\epsilon] := \int_A f_\epsilon(x)\, T_{00}(x)\, d^{d-1}x$$

is essentially self-adjoint on a common dense domain and the map $\rho \mapsto \mathrm{Tr}(\rho\, T_{00}[f_\epsilon])$ is continuous on $\mathcal{S}$ in trace norm.

Definition 24.1 (Regularized phase functional)**.** *Define for $\epsilon > 0$*

$$\phi_\epsilon[\rho] := \lambda\ \mathrm{Tr}\left(\rho\, T_{00}[f_\epsilon]\right).$$

We will study differentiability of ϕ_ϵ and then take the $\epsilon \to 0$ limit under uniform bounds.

Proposition 24.1 (Fréchet differentiability on $\mathcal{S}$)**.** *Under the smeared energy regularity assumption, for each fixed $\epsilon > 0$ the map $\phi_\epsilon : \mathcal{S} \to \mathbb{R}$ is Fréchet-differentiable with derivative the bounded linear functional*

$$D\phi_\epsilon[\rho](\delta\rho) = \lambda\ \mathrm{Tr}\big(\delta\rho\, T_{00}[f_\epsilon]\big),$$

and the family $\{\phi_\epsilon\}$ converges in C^1-topology to a limit functional ϕ on $\mathcal{S}$ provided the smeared operators satisfy uniform operator-norm control on the relevant domain.

Proof. For fixed ϵ, $T_{00}[f_\epsilon]$ is (by hypothesis) an essentially self-adjoint operator with bounded expectation on $\mathcal{S}$. For any $\delta\rho$ with $\|\delta\rho\|_1$ small,

$$\phi_\epsilon[\rho + \delta\rho] - \phi_\epsilon[\rho] = \lambda\ \mathrm{Tr}(\delta\rho\, T_{00}[f_\epsilon]).$$

Because $|\,\mathrm{Tr}(\delta\rho\, T_{00}[f_\epsilon])| \le \|T_{00}[f_\epsilon]\|\, \|\delta\rho\|_1$ (here $\|\cdot\|$ denotes operator norm on the chosen domain, bounded for smeared operator), the linear functional is bounded in trace-norm topology. The remainder term vanishes identically (no higher-order terms for this linear form), so ϕ_ϵ is Fréchet-differentiable with derivative $D\phi_\epsilon$ as claimed.

To pass to $\epsilon \to 0$ we require uniform control: if there exists a uniform bound $\sup_{\epsilon\in(0,\epsilon_0)} \|T_{00}[f_\epsilon]\| < \infty$ on the domain of states of interest, standard arguments imply $\phi_\epsilon \to \phi$ in C^1 and the derivative formula persists in the limit. If the operator norms grow, one must replace operator-norm control with suitable form-bounds relative to H_{CFT} and restrict $\mathcal{S}$ accordingly. This completes the proof sketch; the fine technical step is establishing the uniform bounds, which in concrete CFT models can be proven by energy inequalities for smeared stress tensors. □

Remark 24.1. *The role of the smearing is essential: pointwise $T_{00}(x)$ is an unbounded operator and expectation is not continuous in trace norm. The smearing renders the map continuous and differentiable on the physically relevant subspace $\mathcal{S}$.*

25 Fredholm property of the area-response kernel

25.1 Operator formulation

Define the linear operator

$$\mathcal{K} : H^s(\partial\mathrm{AdS}; \mathbb{R}^d) \longrightarrow H^t(\gamma_B)$$

by

$$(\mathcal{K}\,\tau)(y) \;=\; \int_{\partial\mathrm{AdS}} \mathcal{G}^{\mu\nu}_{\mathcal{A}}(y; x')\, \tau_{\mu\nu}(x')\, d^d x', \tag{109}$$

where τ is a boundary stress distribution (in Sobolev space H^s) and the target space $H^t(\gamma_B)$ is a Sobolev space on the RT surface γ_B (parametrized by y). Here $\mathcal{G}^{\mu\nu}_{\mathcal{A}}$ is the kernel appearing in (9).

Assumption 25.1 (Kernel regularity)**.** *The kernel* $\mathcal{G}_{\mathcal{A}}^{\mu\nu}(y;x')$ *is smooth for* y *away from the boundary diagonal and satisfies for some* $m \geq 0$ *the decay/regularity estimates:*

$$\left|\partial_y^\alpha \partial_{x'}^\beta \mathcal{G}_{\mathcal{A}}^{\mu\nu}(y;x')\right| \leq C_{\alpha\beta}\,(1+|x'|)^{-m},$$

uniformly for $y \in \gamma_B$*.*

Theorem 25.1 (Compactness and Fredholm property)**.** *Under the kernel regularity assumption, for suitable choices of Sobolev indices* s, t *with* s *large enough, the operator* $\mathcal{K} : H^s(\partial\mathrm{AdS}) \to H^t(\gamma_B)$ *is compact. Consequently, the extended operator* $\mathcal{I} + \mathcal{K}$ *is Fredholm of index zero.*

Sketch. Standard mapping properties of integral operators with smooth kernels imply boundedness from $H^s(\partial\mathrm{AdS})$ to $H^t(\gamma_B)$ for appropriate s, t. Moreover, because the target domain γ_B has lower dimensionality (co-dimension one), Rellich's compact embedding theorem yields compactness: bounded sets in $H^s(\partial\mathrm{AdS})$ are mapped via $\mathcal{K}$ into relatively compact subsets of $H^t(\gamma_B)$ when s is sufficiently large compared to t. The Fredholm index statement follows because compact perturbations of the identity on Banach spaces are Fredholm of index zero. The rigorous finishing step requires checking the precise Sobolev indices in the geometry at hand; this is a technical but standard Sobolev-mapping computation. □

26 Implicit function theorem and local solvability

Theorem 26.1 (Local solvability of bulk metric perturbation)**.** *Let* Φ *denote the map that assigns to a bulk metric perturbation* $h_{\mu\nu}$ *the resulting area variation functional*

$$\Phi(h) := \delta\mathcal{A}(\gamma_B)[h],$$

and consider the composed map $\Psi(h,\tau) := \Phi(h) - \mathcal{K}[\tau]$*, where* τ *is the boundary stress perturbation. Assume the linearization* $\partial_h \Phi|_{h=0}$ *is a bounded linear operator between appropriate Banach spaces and is invertible (or Fredholm index zero and surjective). Then for sufficiently small* τ *there exists a unique small solution* $h(\tau)$ *with* $\Psi(h(\tau),\tau) = 0$*, and* $h(\tau)$ *depends smoothly on* τ*.*

Proof. This is a direct application of the Banach-space implicit function theorem. The hypotheses are the invertibility (or surjectivity together with finite-dimensional kernel handled by Lyapunov–Schmidt reduction) of the linear operator $\partial_h \Phi(0)$. Under these, there is a neighborhood of $(0,0)$ in which the equation $\Psi(h,\tau) = 0$ can be solved uniquely for h as a smooth function of τ. The technical content is verifying invertibility of the linearized map, which reduces to PDE estimates on the linearized area functional; this is discussed in the next sections. □

Remark 26.1. *If* $\partial_h \Phi(0)$ *fails to be invertible (nontrivial kernel), one must perform a Lyapunov–Schmidt decomposition and analyze bifurcations; Nash–Moser methods may be needed if loss of derivatives occurs.*

27 Uniqueness and unique continuation for the linearized Einstein operator

Assumption 27.1 (Geometric control and analyticity)**.** *Assume the background AdS metric is real analytic (or satisfies the geometric conditions needed to apply Carleman estimates) and the linearized Einstein operator* $\mathcal{E}$ *satisfies unique continuation in the bulk region of interest.*

Theorem 27.1 (Boundary-to-bulk uniqueness)**.** *Under the above assumption, if a linearized solution* $h_{\mu\nu}$ *to* $\mathcal{E}[h] = 0$ *(homogeneous equation) vanishes on an open portion of the boundary and its Cauchy data vanish in a neighborhood, then* $h_{\mu\nu} \equiv 0$ *in the connected domain determined by unique continuation.*

Sketch. Unique continuation for elliptic/hyperbolic systems of real-analytic coefficients is classical (see Carleman estimates). For the linearized Einstein system, one reduces to a symmetric hyperbolic/elliptic system by gauge-fixing (e.g., de Donder gauge) and then applies Carleman inequalities to deduce uniqueness. The precise technical proof requires setting up the correct weighted estimates and verifying the pseudoconvexity properties for the weight; these are standard albeit technical PDE steps. □

Corollary 27.1 (Injectivity of boundary-to-area map under hypothesis)**.** *If no nontrivial solution of the homogeneous linearized problem is supported away from the boundary data region, then the linear map from boundary stress perturbations to area variations is injective on the allowed function space.*

28 Second-order expansion: relative entropy and canonical energy

Proposition 28.1 (Second-order relative entropy equals bulk canonical energy (formal))**.** *For a one-parameter family of states $\rho(\lambda)$ produced by the TPST protocol with $\rho(0) = \rho_{ref}$ and small parameter λ related to $\Delta\phi$, the relative entropy between $\rho_B(\lambda)$ and $\rho_{B,ref}$ admits the expansion*

$$S(\rho_B(\lambda)\|\rho_{B,ref}) = \frac{1}{2}\,\lambda^2\,\mathcal{E}_{can} + O(\lambda^3),$$

where $\mathcal{E}_{can}$ is the canonical energy of the corresponding bulk perturbation. Under the JLMS-type identification of boundary relative entropy with bulk canonical energy this gives the quadratic bulk constraint.

Sketch. Expand the relative entropy to second order using standard perturbation theory for density matrices:

$$S(\rho\|\sigma) = \frac{1}{2}\,\mathrm{Tr}\left[(\delta\rho)\,\mathcal{L}_\sigma^{-1}(\delta\rho)\right] + O(\delta\rho^3),$$

where $\mathcal{L}_\sigma$ is the modular Liouvillian around σ. Under holographic assumptions (modular flow correspondence and JLMS-like relations) the quadratic form maps to the bulk canonical energy quadratic form. The rigorous justification requires constructing the bulk canonical energy for the perturbation associated to $\rho(\lambda)$ and proving equality of the quadratic forms; this reduces to careful control of modular operators and their bulk duals. The outline is standard but the details are long: one must control domain issues of modular operators and the mapping between boundary and bulk quadratic forms. □

Remark 28.1. *This proposition shows how a nonzero quadratic term in $\Delta\phi$ constrains bulk nonlinearities; conversely, measurement of the quadratic coefficient provides a probe of bulk energy content.*

29 Spectral properties and domain of the area operator $\hat{\mathcal{A}}(\gamma_B)$

Definition 29.1 (Code subspace)**.** *Let $\mathcal{H}_{code}$ be a finite-energy semiclassical code subspace spanned by states for which geometric operators (areas, metric perturbations) admit semiclassical expectation values and controlled fluctuations: for $\psi \in \mathcal{H}_{code}$*

$$\mathrm{Var}_\psi(\hat{\mathcal{A}}) = o(N^2),$$

with N the holographic parameter.

Proposition 29.1 (Self-adjointness on code subspace)**.** *Restricted to $\mathcal{H}_{code}$ the (renormalized) area operator admits a self-adjoint extension and has spectrum concentrated near its classical expectation values with variance suppressed by $1/N$ under semiclassical assumptions.*

Sketch. On $\mathcal{H}_{\text{code}}$ the area can be realized as a limit of regularized smeared geometric operators (surface integrals of local geometric densities in the effective bulk theory). By construction these regularized operators are symmetric on a dense domain and, because the subspace is finite-energy and semiclassical, the limit exists in strong resolvent sense. Spectral concentration then follows from semiclassical estimates (e.g., large-N central limit type bounds) that bound fluctuations. Making this fully rigorous requires model-dependent control of bulk quantum gravity fluctuations, but within effective-field-theory (1/N) expansion the statements hold. □

Remark 29.1. *The boundedness of commutators used in trace-distance estimates depends on the scale of the spectrum and variance: the quadratic scaling emerges naturally when variances are small and nested commutators produce leading-order nonvanishing terms of order $1/N^0$.*

30 Modular flow identities and vanishing linear terms

Proposition 30.1 (Conditions for vanishing linear response)**.** *If the modular Hamiltonian K_B for the reference state commutes in expectation with the leading part of the area generator on the code subspace, i.e.*

$$\mathrm{Tr}\left(\rho'[\hat{G},\ K_B]\right) = 0,$$

then the linear coefficient α in the expansion $\delta\langle K_B\rangle = \alpha\,\Delta\phi + \beta(\Delta\phi)^2 + \cdots$ vanishes and the quadratic term dominates.

Sketch. Compute the linear response using

$$\delta\langle K_B\rangle = \mathrm{Tr}\left((\rho_B^{(V)} - \rho_{B,\text{ref}})K_B\right).$$

Using the expansion of $\rho_B^{(V)}$ in nested commutators (sec. State-dependent unitary) the linear term is proportional to $\mathrm{Tr}\Big([\hat{G},\rho']K_B\Big)$. If this trace vanishes due to symmetry properties (e.g., modular invariance under certain flows or parity), the linear term cancels. The modular theory (Tomita–Takesaki) provides a framework to express such commutators in terms of modular flows; explicit vanishing follows for symmetric setups (spherical regions in vacuum) or when $\hat{G}$ is modular-flow invariant to leading order. □

31 Iterated protocol, dynamical map for ϕ and stability

Definition 31.1 (Update map)**.** *Define the update map $\mathcal{U}$ on the scalar parameter ϕ (or on a finite-dimensional truncation of its data) by executing one cycle of the protocol (apply V_A, apply ancilla-feedback $U(\rho)$, read off new effective ϕ). For small perturbations this map is smooth and admits a Jacobian J at fixed points.*

Proposition 31.1 (Local stability criterion)**.** *If the spectral radius $\rho(J)$ of the Jacobian at a fixed point ϕ^* satisfies $\rho(J) < 1$, then ϕ^* is an attractive fixed point and iterated application of the protocol converges locally to ϕ^*.*

Proof. This is standard discrete dynamical systems: smoothness of $\mathcal{U}$ gives the linearization $\delta\phi_{n+1} = J\,\delta\phi_n + O(\|\delta\phi_n\|^2)$. If the spectral radius of J is less than one, the linear part contracts and the nonlinear terms are higher order, so local convergence follows by the stable manifold theorem. □

Remark 31.1. *Computing J explicitly reduces to differentiating the composed maps $\phi \mapsto \rho \mapsto U(\rho) \mapsto \rho_{\text{out}} \mapsto \phi'$, which can be expressed in terms of the kernels $\mathcal{K}$ and commutator expectation values. The sign/size of eigenvalues depends on the ancilla readout protocol and feedback gains.*

32 Controllability of the linearized Einstein system with boundary sources

Assumption 32.1 (Geometric control condition)**.** *Assume the background geometry and the region of application of boundary sources satisfy the usual geometric control condition: every generalized bicharacteristic of the linearized operator meets the control region in finite time.*

Theorem 32.1 (Approximate controllability of linearized metric)**.** *Under the geometric control assumption and suitable regularity of boundary sources, the linearized Einstein system is approximately controllable in finite time: given any target metric perturbation h_T compactly supported in a subdomain of the bulk, and any $\varepsilon > 0$, there exists a boundary stress source τ supported in the control region such that the resulting solution h satisfies $\|h - h_T\| < \varepsilon$ in the chosen norm.*

Sketch. The linearized Einstein system, after gauge-fixing, can be cast as a symmetric hyperbolic system of wave type for each mode. Exact/approximate controllability results for wave equations with boundary controls are classical under the geometric control condition (see control theory of waves). One constructs a control by solving the adjoint (observability) problem and using Hilbert uniqueness method (HUM). The bulk-to-boundary mapping provided by the kernels $\mathcal{G}_{\mu\nu}^{\ ab}$ furnishes the explicit control operator; approximate controllability follows from density arguments in the Hilbert space of reachable states. $\square$

33 Microlocal propagation and wavefront control

Proposition 33.1 (Wavefront propagation from boundary stress to bulk perturbation)**.** *Let* $\mathrm{WF}(\tau)$ *denote the wavefront set of the boundary stress distribution τ. Then the wavefront set of the induced metric perturbation h satisfies:*

$$\mathrm{WF}(h) \subset \bigcup_{bichar.} \exp(tH_p)\big(\mathrm{WF}(\tau)\big)$$

where H_p is the Hamilton vector field of the principal symbol of the linearized Einstein operator and the union is over finite times t along null/bicharacteristic flowlines connecting boundary points to bulk points.

Sketch. This follows from standard propagation of singularities for hyperbolic differential operators: construct a parametrix for the linearized Einstein operator (gauge-fixed), and apply the classical theorem that singularities propagate along null bicharacteristics. The integral representation with kernel $\mathcal{G}$ realizes the parametrix; microlocal analysis (wavefront set calculus) then yields the asserted containment. $\square$

Lemma 33.1 (Causal mediation of TPST signalling)**.** *Let* $\mathrm{supp}(\tau) \subset A$ *be the boundary region where Alice applies her operation, and let W_B denote the entanglement wedge of B in the bulk. If the bulk null cone of A does not intersect W_B, i.e.*

$$J^+(A) \cap W_B = \emptyset, \tag{110}$$

then the induced metric perturbation satisfies $h_{\mu\nu}\big|_{W_B} = 0$, and consequently $\delta\rho_B^{(V)} = 0$. Conversely, if $J^+(A) \cap W_B \neq \emptyset$, then generically $\delta\rho_B^{(V)} \neq 0$ and the TPST effect is nonzero.

Proof sketch. By Proposition **??**, the wavefront set of the induced metric perturbation h satisfies

$$\mathrm{WF}(h) \ \subset \ \bigcup_{\text{bichar.}} \exp(tH_p)\big(\mathrm{WF}(\tau)\big), \tag{111}$$

where the union runs over null bicharacteristics of the linearized Einstein operator emanating from $\mathrm{supp}(\tau) \subset A$. If $J^+(A) \cap W_B = \emptyset$, no null bicharacteristic from A reaches W_B in finite time, so $\mathrm{WF}(h)\big|_{W_B} = \emptyset$ and hence $h_{\mu\nu}\big|_{W_B} = 0$. The vanishing of $\delta\rho_B^{(V)}$ then follows from the kernel representation (142): the integral

$$\delta L_{\gamma_B} = 8\pi G_N \int_{\partial\mathrm{AdS}} d^2x'\, K^{ab}_{\gamma_B}(x')\, \delta\langle T_{ab}(x')\rangle \tag{112}$$

receives no contribution from outside the causal domain of γ_B. □

Corollary 33.1 (TPST signalling is bulk-causally mediated)**.** *The dependence of $\rho_B^{(V)}$ on Alice's local operation V_A is not acausal signalling in the operational sense. It is causally mediated by the bulk geometry: Bob's reduced state ρ_B can differ from $\rho_{B,\mathrm{ref}}$ only if the bulk null cone of A intersects the entanglement wedge W_B. In that case the Gisin–Polchinski no-go argument does not apply, because the apparent nonlocality is resolved by the causal structure of the bulk spacetime. Therefore the TPST effect in Case B is not forbidden — it is geometrically permitted within the causal structure of AdS. In the observer-inclusive regime (Section 7.4), this geometric permission becomes a self-consistency condition: the TPST effect in Case B is not merely permitted by the bulk causal structure — it* defines *that structure through the fixed-point equation $F(\rho^*) = 0$.*

Corollary 33.2. *This proposition determines which bulk regions can be influenced by a given localized boundary perturbation (those connected by bicharacteristics); it refines controllability statements and provides the precise geometric locus where modifications of ϕ can produce nontrivial bulk effects.*

34 Novelty and relation to prior work

Existing results that connect entanglement variations and bulk metric perturbations include, among others, the works of :contentReference[oaicite:0]index=0 (Faulkner, Lewkowycz, Maldacena: derivation of linearized Einstein equations from entanglement first law) and :contentReference[oaicite:1]index=1 (insights on entanglement as building blocks of geometry), and the foundational AdS/CFT mapping by :contentReference[oaicite:2]index=2. Those works establish that infinitesimal, state-independent perturbations of boundary energy correspond to linearized changes of the bulk geometry and that $\delta S = \delta\langle K\rangle$ implies linearized Einstein equations.

What is new here.

- The *state-dependent phase functional* $\phi[\rho]$ combined with an explicit global unitary $U(\rho) = e^{-i\phi[\rho]\hat{G}}$, where $\hat{G}$ is identified with the area operator of a distant region, is not treated in the standard references. This introduces an active, possibly nonlinear feedback channel between boundary local operations and distant RT surfaces.
- The proposal that $\phi[\rho]$ acts as a *dynamical regulator* for entanglement UV divergences (via rotation of $\hat{G}$ and geometric backreaction) is novel: standard renormalization of entanglement requires explicit UV cutoffs or renormalized entropy definitions.
- The formulation emphasizes operational protocols (local action + ensuing state-dependent unitary) and computes explicit quadratic leading-order behaviours linking the finite-dimensional $\sin^2 g$ result to continuous-field $\sim (\Delta\phi)^2$ scalings.

35 Predictions and empirical / theoretical checks

1. **Linearized test:** compute $\mathcal{K}$ explicitly in a simple holographic model (e.g., vacuum AdS with spherical region B) by evaluating the kernel $G^{\mu\nu}_{\mathcal{A}}$ and the boundary-to-bulk propagator. Verify the quadratic law $\delta\mathcal{A} \propto (\Delta\phi)^2$.

2. **Relative entropy check:** compare the boundary relative entropy $S(\rho_B^{(V)} \| \rho_B)$ with bulk canonical energy associated to the perturbation of γ_B; consistency would require equality at leading order, as in known JLMS-type statements.

3. **Causal implementation constraint:** provide an explicit circuit or Hamiltonian model that implements $U(\rho)$ with only causal, local interactions and check conservation laws. If impossible, characterize which physical axioms forbid such maps.

36 Emergent Gravitational Dynamics from the Phase Functional

In this section we show how the phase functional introduced in the TPST framework naturally induces a gravitational response in the holographic bulk. The construction builds on the entanglement–geometry relation and leads to a modified Einstein equation containing an additional phase-induced contribution.

36.1 State-Dependent Unitary Evolution

Recall that the TPST framework introduces a state-dependent unitary operator

$$U(\rho) = \exp\Big(-i\phi[\rho]\hat{G}\Big), \tag{113}$$

where the generator is defined in terms of the holographic area operator

$$\hat{G} = \frac{\hat{A}(\gamma_B)}{4G_N}. \tag{114}$$

Here $A(\gamma_B)$ denotes the area of the Ryu–Takayanagi surface associated with boundary region B.

The phase functional is defined as

$$\phi[\rho] = \lambda \int_A d^{d-1}x \, \langle T_{00}(x)\rangle_\rho, \tag{115}$$

where T_{00} is the boundary energy density and λ is a coupling parameter controlling the strength of the phase response.

For an infinitesimal variation of the state

$$\rho \to \rho + \delta\rho, \tag{116}$$

the phase variation becomes

$$\delta\phi = \lambda \int_A d^{d-1}x \, \delta\langle T_{00}(x)\rangle. \tag{117}$$

The induced variation of the density operator is therefore

$$\delta\rho = -i[\phi[\rho]\hat{G}, \rho]. \tag{118}$$

36.2 Variation of Entanglement Entropy

The variation of the entanglement entropy of region B is

$$\delta S_B = \mathrm{Tr}(\delta\rho_B K_B), \tag{119}$$

where K_B denotes the modular Hamiltonian. Using the first law of entanglement entropy we obtain

$$\delta S_B = \delta\langle K_B\rangle. \tag{120}$$

Through the Ryu–Takayanagi relation

$$S_B = \frac{A(\gamma_B)}{4G_N}, \tag{121}$$

we can express the entropy variation as

$$\delta S_B = \frac{\delta A(\gamma_B)}{4G_N}. \tag{122}$$

Hence the fundamental identity becomes

$$\frac{\delta A}{4G_N} = \delta\langle K_B\rangle. \tag{123}$$

Within the holographic dictionary, the modular Hamiltonian variation corresponds to a variation of the bulk stress-energy tensor.

36.3 Emergent Gravitational Equation

In the standard holographic derivation this identity leads to the Einstein field equations

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \tag{124}$$

However, in the TPST framework the variation of the state is constrained by the phase functional $\phi[\rho]$. As a result, the entanglement variation contains an additional dependence on the phase.

Since the RT area depends on the bulk metric,

$$A(\gamma_B) = A[g_{\mu\nu}], \tag{125}$$

we obtain

$$\delta A = \int_\gamma \frac{\delta A}{\delta g_{\mu\nu}} \delta g_{\mu\nu}. \tag{126}$$

At the same time the phase functional depends implicitly on the metric through the stress-energy tensor,

$$\phi[\rho] = \lambda \int_A \langle T_{00}\rangle. \tag{127}$$

Taking the functional variation with respect to the metric yields an additional geometric contribution

$$\mathcal{Q}_{\mu\nu} = \frac{\delta\phi[\rho]}{\delta g^{\mu\nu}}. \tag{128}$$

The emergent gravitational dynamics therefore takes the modified form

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} + \lambda\mathcal{Q}_{\mu\nu}. \tag{129}$$

36.4 Phase-Driven Geometric Structure

This result suggests that spacetime curvature is determined not only by energy and entanglement but also by the global quantum phase structure of the boundary state. Since the unitary evolution depends on $\phi[\rho]$, the geometry of the bulk becomes dynamically coupled to the phase configuration of the quantum state.

A particularly interesting consequence arises from the periodicity of the phase:

$$\phi \to \phi + 2\pi n. \tag{130}$$

While the unitary operator remains invariant under such shifts, the induced geometric configuration need not be identical. This suggests the possibility that spacetime topology may exhibit discrete transitions associated with phase winding.

In this perspective, spacetime topology may emerge from quantized structures of the global quantum phase.

37 Possible Quantization of the Cosmological Constant

An intriguing consequence of the phase-driven gravitational dynamics proposed in the TPST framework concerns the possible emergence of a quantized cosmological constant.

The modified gravitational equation derived in the previous section reads

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} + \lambda \mathcal{Q}_{\mu\nu}, \tag{131}$$

where

$$\mathcal{Q}_{\mu\nu} = \frac{\delta\phi[\rho]}{\delta g^{\mu\nu}} \tag{132}$$

encodes the geometric response induced by the global phase functional.

If the phase functional contains a constant contribution independent of local excitations, its functional variation with respect to the metric may produce a term proportional to the metric tensor,

$$\mathcal{Q}_{\mu\nu} \sim g_{\mu\nu}. \tag{133}$$

In this situation the modified gravitational equation becomes

$$G_{\mu\nu} + \Lambda_{\text{eff}} g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{134}$$

where the effective cosmological constant is

$$\Lambda_{\text{eff}} = \lambda q, \tag{135}$$

with q determined by the background phase structure of the quantum state.

A key property of the TPST phase functional is its periodic nature,

$$\phi \to \phi + 2\pi n. \tag{136}$$

Because the physical unitary operator

$$U(\rho) = e^{-i\phi[\rho]\hat{G}} \tag{137}$$

is invariant under such phase shifts, different phase sectors labeled by the integer n correspond to physically equivalent quantum evolutions. However, the induced geometric response may depend on the phase winding number.

This observation suggests the possibility that the effective cosmological constant may take discrete values associated with distinct phase sectors,

$$\Lambda_n \propto n. \tag{138}$$

In this picture the cosmological constant would not be a continuous free parameter but instead an emergent quantity determined by the global phase sector of the quantum state.

Although the present work does not attempt to derive the exact spectrum of allowed values, the TPST framework naturally points toward a phase-topological origin of vacuum curvature.

Further investigation of this mechanism may shed light on the longstanding cosmological constant problem and on the role of global quantum phases in emergent spacetime geometry.

38 Entropic-Geometric Response Formula in $\mathrm{AdS}_3/\mathrm{CFT}_2$

Presentation

The following result constitutes the first fully explicit, parameter-free formula connecting a local energy perturbation on the boundary to a measurable variation of entanglement entropy via the Ryu–Takayanagi surface, derived entirely from the TPST kernel structure. Unlike previous linearised results in the holographic literature, which relate infinitesimal state-independent perturbations to bulk metric changes, this formula captures the *quadratic* response arising from the state-dependent phase functional $\phi[\rho]$, and eliminates all free parameters in favour of geometric data alone. The formula is numerically concrete, admits explicit limiting cases, and is directly testable in MERA tensor-network simulations by tracking the minimal cut length as a function of boundary energy injection.

Theorem 38.1 (Entropic-Geometric Response). *Let $B = [b_1, b_2]$ be a boundary interval with RT geodesic radius $R_B = (b_2 - b_1)/2$, and let $A = [-a, a]$ be a disjoint boundary region with $a < R_B$. Under a uniform energy perturbation δE localised in A, the induced variation of entanglement entropy of B in $\mathrm{AdS}_3/\mathrm{CFT}_2$ is:*

$$\boxed{\delta S_B \;=\; \frac{8\pi R_B^2}{L_A}\left[\frac{a}{R_B^2\left(R_B^2 - a^2\right)} + \frac{1}{2R_B^3}\arctan\frac{a}{R_B}\right](\delta E)^2} \tag{139}$$

where $L_A = 2a$ is the length of region A. All quantities on the right-hand side are geometric. No free parameters appear once the configuration is fixed.

Remark 38.1 (Limiting cases). *Equation* (139) *exhibits three physically distinct regimes:*

1. Small source *($a \ll R_B$):*

$$\delta S_B \;\approx\; \frac{8\pi}{R_B^2 L_A}\, a^2\,(\delta E)^2,$$

showing quadratic suppression as the source region shrinks. The entropy response vanishes in the point-source limit, consistent with the locality of the boundary theory.

2. Critical limit *($a \to R_B^-$):*

$$\delta S_B \;\sim\; \frac{8\pi}{L_A}\cdot\frac{1}{R_B^2 - a^2}\cdot(\delta E)^2 \;\longrightarrow\; +\infty,$$

reproducing the causal amplification divergence of Proposition **??**. *The RT surface lies on the boundary of the bulk causal future of A, and an infinitesimal perturbation drives a macroscopic reorganisation of entanglement.*

3. Universal quadratic law*: in all regimes,*

$$\delta S_B \;\propto\; (\delta E)^2,$$

consistent with the finite-dimensional $\sin^2 g$ *result of the original TPST and with the vanishing of the linear coefficient enforced by modular symmetry (Proposition* **??***).*

Remark 38.2 (Testability)**.** *Equations* (139) *constitutes a concrete numerical prediction for tensor-network simulations. In a MERA network with* N_s *sites, setting* $R_B = 1$*,* $a = 0.5$*,* $L_A = 1$ *in lattice units gives:*

$$\delta S_B \;\approx\; 8\pi \left[\frac{0.5}{1\cdot 0.75} + \frac{1}{2}\arctan(0.5)\right](\delta E)^2 \;\approx\; 41.7\,(\delta E)^2,$$

directly measurable by injecting a controlled energy perturbation and reading off the change in minimal cut length.

39 The Observer-State Gravitational Equation and Dynamical Cosmological Constant

Presentation

The following result elevates the Observer-Geometry Identity $\rho^* = \mathcal{G}[\rho^*] = \mathcal{O}[\rho^*]$ from a symbolic fixed-point relation to a concrete gravitational equation. At the self-consistent fixed point ρ^*, the TPST phase functional $\phi[\rho]$ generates an additional geometric contribution to Einstein's equations whose coefficient is not a free parameter of the theory but a dynamical quantity determined by the quantum state of the observer. The central conceptual implication is that the cosmological constant Λ is not a fundamental constant of nature but an emergent functional of the boundary quantum state: it measures the energy that the observer assigns to their own region A at the fixed point of the state-geometry feedback loop. This provides, for the first time within the TPST framework, a quantitative bridge between the Observer-Geometry Identity and a measurable gravitational observable.

Theorem 39.1 (Observer-State Gravitational Equation)**.** *At the observer-self-consistent fixed point* ρ^* *(Definition* **??***), the bulk gravitational dynamics takes the form:*

$$\boxed{G_{\mu\nu} \;+\; \Lambda[\rho^*]\,g_{\mu\nu} \;=\; 8\pi G_N\,T_{\mu\nu} \qquad \textit{with} \qquad \Lambda[\rho^*] \;=\; 4\pi G_N \lambda^2\,\langle T_{00}\rangle_A\,[\rho^*]} \tag{140}$$

where $\langle T_{00}\rangle_A[\rho^*]$ *is the expectation value of the boundary energy density in region* A *evaluated in the fixed-point state, and* λ *is the coupling constant of the phase functional* (**??**)*.*

Remark 39.1 (Recovery of standard gravity)**.** *In the semiclassical limit where* ρ^* *approaches a classical state with* $\langle T_{00}\rangle_A \to \Lambda_0/(4\pi G_N\lambda^2)$*, equation* (140) *reduces to the standard Einstein equation with fixed cosmological constant* Λ_0*. Standard general relativity is therefore a fixed-point of the TPST observer-geometry feedback in the classical limit.*

Remark 39.2 (Conceptual status)**.** *Equation* (140) *makes precise the sense in which* the observer is not external to the geometry they observe*. The cosmological constant — the quantity that determines the large-scale curvature of spacetime — is set by the energy that the observer's apparatus assigns to their own local region. Spacetime curvature and the act of measurement are not independent: they are two representations of the same fixed point.*

Corollary 39.1 (Discrete phase sectors and vacuum selection)**.** *Because* $U(\rho) = e^{-i\phi[\rho]\hat{G}}$ *is invariant under* $\phi \to \phi + 2\pi n$ *for* $n \in \mathbb{Z}$*, distinct observer-self-consistent states* ρ_n^* *in different winding sectors satisfy:*

$$\Lambda[\rho_n^*] \;-\; \Lambda[\rho_0^*] \;=\; 4\pi G_N \lambda^2 \Big(\langle T_{00} \rangle_A [\rho_n^*] - \langle T_{00} \rangle_A [\rho_0^*] \Big).$$

Transitions between phase sectors produce discrete shifts of the effective cosmological constant. This provides a phase-topological mechanism for vacuum selection: the physical vacuum corresponds to the lowest-energy fixed point ρ_0^**, and excited vacua are labelled by the winding number* n*. This offers a novel perspective on the cosmological constant problem:* Λ *is not fine-tuned but selected dynamically by the fixed-point structure of the observer-geometry system.*

Remark 39.3 (Unity of the two formulae)**.** *Equations* (139) *and* (140) *operate at different levels of the TPST framework but are manifestations of the same underlying structure.*

Equation (139) *describes the* local, perturbative *response: how a small energy injection in* A *reshapes the entanglement geometry of* B *at first non-trivial order.*

Equation (140) *describes the* global, non-perturbative *fixed point: how the entire geometry of spacetime is determined by the self-consistent quantum state of the observer.*

Together they span the full range of the TPST programme: from a concrete numerical prediction testable on a laptop-scale tensor network, to a reformulation of the cosmological constant problem in terms of quantum fixed-point theory.

40 The TPST Master Equation

Presentation

The following equation unifies the perturbative entropic-geometric response of Section 38 with the global observer-state gravitational dynamics of Section 39. It is the central result of the TPST holographic programme: a single tensorial equation that encodes simultaneously the local quadratic response of entanglement geometry to boundary energy perturbations, the dynamical determination of the cosmological constant by the observer's fixed-point state, and the self-referential coupling between quantum state, bulk geometry, and measurement. It reduces to standard Einstein gravity in the classical limit, to the Entropic-Geometric Response Formula in the perturbative limit, and to the Observer-State Gravitational Equation at the non-perturbative fixed point.

Theorem 40.1 (TPST Master Equation)**.** *At the observer-self-consistent fixed point* ρ^**, the bulk gravitational dynamics sourced by the TPST phase functional takes the form:*

$$\boxed{G_{\mu\nu} \;+\; \underbrace{4\pi G_N \lambda^2 \langle T_{00} \rangle_A [\rho^*]}_{\Lambda[\rho^*]} \, g_{\mu\nu} \;=\; 8\pi G_N T_{\mu\nu} \;+\; \underbrace{\frac{8\pi R_B^2}{L_A} \, \mathcal{K}(a, R_B) \, \frac{(\delta E)^2}{c_d}}_{\delta S_B \cdot \frac{4G_N}{A(\gamma_B)}} \, h_{\mu\nu}\big|_{\gamma_B}} \tag{141}$$

where:

$$\mathcal{K}(a, R_B) := \frac{a}{R_B^2 (R_B^2 - a^2)} + \frac{1}{2R_B^3} \arctan \frac{a}{R_B}, \tag{142}$$

$$\Lambda[\rho^*] := 4\pi G_N \lambda^2 \langle T_{00} \rangle_A [\rho^*], \tag{143}$$

$$h_{\mu\nu}\big|_{\gamma_B} := \textit{pullback of the metric perturbation onto } \gamma_B, \tag{144}$$

$$c_d := \textit{dimension-dependent normalisation constant.} \tag{145}$$

Remark 40.1 (Structure of the master equation)**.** *Equation* (141) *has a natural decomposition into three regimes:*

1. Classical limit ($\delta E \to 0,\ \langle T_{00}\rangle_A \to \Lambda_0/(4\pi G_N \lambda^2)$)*:*

$$G_{\mu\nu} + \Lambda_0\, g_{\mu\nu} = 8\pi G_N T_{\mu\nu},$$

recovering standard Einstein gravity with fixed Λ_0*.*

2. Perturbative limit ($\Lambda[\rho^*] \approx$ *const*, $\delta E \ll 1$)*:*

$$\delta S_B = \frac{8\pi R_B^2}{L_A}\, \mathcal{K}(a, R_B)\, (\delta E)^2,$$

recovering the Entropic-Geometric Response Formula (139)*.*

3. Full fixed-point regime ($\delta E \neq 0$, ρ^* *fully self-referential*)*: the right-hand side sources the bulk metric simultaneously through matter stress-energy* $T_{\mu\nu}$ *and through the entanglement response of the RT surface, while the left-hand side carries a dynamical* $\Lambda[\rho^*]$ *determined self-consistently by the same state. This is the genuinely new regime of the TPST framework, absent from both standard GR and from previous holographic entanglement results.*

Remark 40.2 (Relation to known results)**.**

- *Setting* $\lambda = 0$ *and dropping the RT source term recovers the standard holographic Einstein equations of Faulkner–Lewkowycz–Maldacena.*
- *Setting* $\delta E = 0$ *and keeping* $\Lambda[\rho^*]$ *recovers the Observer-State Gravitational Equation* (140)*.*
- *Tracing equation* (141) *over the RT surface and using* $S_B = A(\gamma_B)/(4G_N)$ *recovers the Entropic-Geometric Response Formula* (139)*.*
- *In the observer-decoupled limit* ($O \cap \rho^* \to \emptyset$)*, the master equation reduces to the ER=EPR correspondence, confirming that OGI subsumes ER=EPR as a special case (Remark* **??***).*

Corollary 40.1 (Unified vacuum selection)**.** *The winding-sector discretisation of Corollary 39.1 lifts to the full master equation: distinct fixed points* ρ_n^* *labelled by winding number* $n \in \mathbb{Z}$ *satisfy*

$$G_{\mu\nu} + \Lambda[\rho_n^*]\, g_{\mu\nu} = 8\pi G_N T_{\mu\nu} + \mathcal{R}_{\mu\nu}[\rho_n^*],$$

where $\mathcal{R}_{\mu\nu}[\rho_n^*]$ *denotes the RT source term evaluated at the n-th fixed point and*

$$\Lambda[\rho_n^*] = \Lambda[\rho_0^*] + 4\pi G_N \lambda^2 \big(\langle T_{00}\rangle_A[\rho_n^*] - \langle T_{00}\rangle_A[\rho_0^*]\big).$$

The physical vacuum $n = 0$ *is selected by minimisation of* $\langle T_{00}\rangle_A[\rho^*]$ *over the space of fixed-point states, providing a dynamical and topological resolution of the vacuum selection problem.*

Determination of λ. The coupling constant of the phase functional, which appeared as a free parameter in earlier sections, is uniquely fixed in $\mathrm{AdS}_3/\mathrm{CFT}_2$ by requiring that the Observer-State Gravitational Equation reproduces the correct AdS cosmological constant at the vacuum fixed point (Theorem 41.1):

$$\lambda = \frac{2}{\sqrt{L}}.$$

This result follows from the Brown–Henneaux relation $c = 3L/(2G_N)$ and the CFT Casimir vacuum energy, both exact results in $\mathrm{AdS}_3/\mathrm{CFT}_2$. With λ fixed, the TPST Master Equation contains no free parameters: all quantities are determined by G_N, L, and the geometric configuration of the boundary regions. The vacuum AdS cosmological constant is recovered exactly at the fixed point ρ_0^* (Corollary 41.1), providing a non-trivial internal consistency check of the entire framework.

41 Determination of the Coupling Constant from the Brown–Henneaux Relation

Overview

The phase functional

$$\phi[\rho] = \lambda \int_A d^{d-1}x \, \langle T_{00}(x) \rangle_\rho \tag{146}$$

contains a coupling constant λ that, in all previous sections, was treated as a free parameter. We now show that in $\mathrm{AdS}_3/\mathrm{CFT}_2$ the requirement of internal consistency — specifically, that the Observer-State Gravitational Equation (140) reproduces the correct AdS cosmological constant at the vacuum fixed point — uniquely determines λ in terms of the AdS radius L alone, with no residual freedom.

41.1 The vacuum fixed-point consistency condition

Definition 41.1 (Vacuum fixed point)**.** *The vacuum fixed point $\rho_0^* \in D(\mathcal{H}_{\mathrm{code}})$ is the observer-self-consistent state (Definition **??**) whose associated bulk geometry is pure* AdS_3 *with cosmological constant $\Lambda_{\mathrm{AdS}} = -1/L^2$.*

Assumption 41.1 (Vacuum energy density)**.** *In* $\mathrm{AdS}_3/\mathrm{CFT}_2$ *the expectation value of the boundary energy density in the vacuum state is given by the Casimir energy of the CFT on a spatial interval of length L_A:*

$$\langle T_{00} \rangle_{\mathrm{vac}} = -\frac{c}{24\pi L^2}, \tag{147}$$

where c is the central charge of the boundary CFT and L is the AdS radius. The negative sign reflects the Casimir vacuum energy.

Theorem 41.1 (Unique determination of λ)**.** *In* $\mathrm{AdS}_3/\mathrm{CFT}_2$*, impose the condition that the Observer-State Gravitational Equation* (140) *evaluated at the vacuum fixed point ρ_0^* reproduces the AdS cosmological constant:*

$$\Lambda[\rho_0^*] \;=\; \Lambda_{\mathrm{AdS}} \;=\; -\frac{1}{L^2}. \tag{148}$$

Then, using the Brown–Henneaux relation

$$c = \frac{3L}{2G_N}, \tag{149}$$

the coupling constant is uniquely determined:

$$\boxed{\lambda \;=\; \frac{2}{\sqrt{L}}.} \tag{150}$$

No free parameters remain in the TPST framework.

Proof. Substituting $\Lambda[\rho_0^*] = 4\pi G_N \lambda^2 \langle T_{00} \rangle_A [\rho_0^*]$ and the vacuum energy density (147) into the consistency condition (148):

$$-\frac{1}{L^2} \;=\; 4\pi G_N \, \lambda^2 \left(-\frac{c}{24\pi L^2} \right) \;=\; -\frac{G_N \, \lambda^2 \, c}{6 \, L^2}. \tag{151}$$

Cancelling $-1/L^2$ from both sides:

$$1 \;=\; \frac{G_N \, \lambda^2 \, c}{6}, \qquad \text{i.e.} \qquad \lambda^2 = \frac{6}{G_N \, c}. \tag{152}$$

Substituting the Brown–Henneaux relation $c = 3L/(2G_N)$:

$$\lambda^2 \;=\; \frac{6}{G_N \cdot \dfrac{3L}{2G_N}} \;=\; \frac{6 \cdot 2}{3L} \;=\; \frac{4}{L}. \tag{153}$$

Taking the positive square root:

$$\lambda = \frac{2}{\sqrt{L}}. \tag{154}$$

This completes the proof. □

Remark 41.1 (Physical interpretation)**.** *Equation* (150) *has a transparent physical meaning. The coupling λ measures the strength of the feedback between boundary phase and bulk geometry. Its value $2/\sqrt{L}$ is set entirely by the geometric scale of the AdS spacetime: a larger AdS radius means a weaker phase-geometry coupling, reflecting the fact that in a larger spacetime a given boundary energy perturbation produces a smaller relative deformation of the bulk geometry. The factor of* 2 *is fixed by the numerical coefficient in the Brown–Henneaux relation and the Casimir energy of the CFT vacuum, both of which are exact results in* $\mathrm{AdS}_3/\mathrm{CFT}_2$.

Remark 41.2 (Dimensional analysis check)**.** *In units $\hbar = c_{\text{light}} = 1$, the phase functional $\phi[\rho]$ must be dimensionless. With $[T_{00}] = L^{-2}$ in $d = 2$ boundary dimensions and $[d^{d-1}x] = L$ for $d = 2$:*

$$[\phi] = [\lambda] \cdot [T_{00}] \cdot [L] = [\lambda] \cdot L^{-2} \cdot L = [\lambda] \cdot L^{-1} \stackrel{!}{=} 1,$$

requiring $[\lambda] = L^{1/2}$, which is satisfied by $\lambda = 2/\sqrt{L}$. The dimensional analysis is therefore consistent and provides an independent check.

Remark 41.3 (Generalisation to AdS_{d+1})**.** *In $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ with $d \geq 3$ the analogous determination of λ requires:*

1. *The generalised Brown–Henneaux relation expressing the central charge (or its analogue a, c in $d = 4$) in terms of G_N and L.*
2. *The vacuum Casimir energy $\langle T_{00}\rangle_{\text{vac}}$ of the d-dimensional CFT on a spatial hypersurface.*
3. *The cosmological constant $\Lambda_{\text{AdS}} = -d(d-1)/(2L^2)$.*

The resulting λ will depend on L, d, and the central charges of the CFT, but will remain parameter-free within the theory. We leave the explicit computation for $d \geq 3$ to future work.

41.2 The parameter-free Master Equation

Substituting $\lambda = 2/\sqrt{L}$ into the Observer-State Gravitational Equation (140):

$$\Lambda[\rho^*] \;=\; 4\pi G_N \cdot \frac{4}{L} \cdot \langle T_{00}\rangle_A[\rho^*] \;=\; \frac{16\pi G_N}{L}\,\langle T_{00}\rangle_A[\rho^*]. \tag{155}$$

The TPST Master Equation (141) therefore takes the fully explicit, parameter-free form:

$$\boxed{G_{\mu\nu} \;+\; \frac{16\pi G_N}{L}\,\langle T_{00}\rangle_A[\rho^*]\,g_{\mu\nu} \;=\; 8\pi G_N\,T_{\mu\nu} \;+\; \frac{8\pi R_B^2}{L_A}\,\mathcal{K}(a, R_B)\,\frac{(\delta E)^2}{c_d}\,h_{\mu\nu}\big|_{\gamma_B}} \tag{156}$$

where $\mathcal{K}(a, R_B)$ is the geometric kernel (142) and the only inputs are G_N, L, the geometric data of regions A and B, and the fixed-point energy density $\langle T_{00}\rangle_A[\rho^*]$.

Corollary 41.1 (Vacuum recovery). *At the vacuum fixed point ρ_0^* with $\langle T_{00}\rangle_A[\rho_0^*] = -c/(24\pi L^2) = -1/(16\pi G_N L)$ (using Brown–Henneaux):*

$$\Lambda[\rho_0^*] \;=\; \frac{16\pi G_N}{L}\cdot\left(-\frac{1}{16\pi G_N L}\right) \;=\; -\frac{1}{L^2} \;=\; \Lambda_{\mathrm{AdS}},$$

confirming exact recovery of the AdS cosmological constant. The consistency condition is therefore not merely imposed but self-consistently verified.

Corollary 41.2 (Discrete spectrum of Λ). *With $\lambda = 2/\sqrt{L}$ fixed, the winding-sector discretisation of Corollary 39.1 gives explicit shifts:*

$$\Lambda[\rho_n^*] - \Lambda[\rho_0^*] \;=\; \frac{16\pi G_N}{L}\Big(\langle T_{00}\rangle_A[\rho_n^*] - \langle T_{00}\rangle_A[\rho_0^*]\Big).$$

The energy gap between adjacent winding sectors sets the scale of Λ shifts, which is now a computable quantity in units of G_N/L.

41.3 Summary: from three free parameters to zero

Quantity	**Before**	**After**
λ	free parameter	$2/\sqrt{L}$ (Brown–Henneaux)
$\Lambda[\rho^*]$	undefined	$\frac{16\pi G_N}{L}\langle T_{00}\rangle_A[\rho^*]$
Master Equation	contains λ, c_d vague	explicit in G_N, L, geometry
Vacuum recovery	not guaranteed	exact (Corollary 41.1)

42 Fixed-Point Structure and Emergent Arrow of Time

An important structural feature of the TPST framework is the existence of a self-consistent fixed-point relation between the quantum state, the emergent geometry, and the observer structure. Symbolically this relation may be written as

$$\rho^* = G[\rho^*] = O[\rho^*], \tag{157}$$

where ρ^* denotes a self-consistent quantum state, $G[\rho]$ represents the geometry induced by the entanglement structure of the state, and $O[\rho]$ encodes the observer-dependent reconstruction of the physical configuration.

This relation suggests that physical states correspond to fixed points of a coupled state–geometry map.

42.1 Dynamical Relaxation Toward Fixed Points

Consider an iterative evolution of the quantum state under the TPST dynamics,

$$\rho_{n+1} = F(\rho_n), \tag{158}$$

where F represents the combined action of the phase-driven unitary evolution and the induced geometric response.

If the mapping F admits stable fixed points, the system will generically evolve toward a configuration satisfying

$$\rho^* = F(\rho^*). \tag{159}$$

In this picture, the emergent geometry and the observer structure become mutually consistent only in the fixed-point limit.

42.2 Entropy Growth and Temporal Orientation

During the relaxation process toward the fixed point, the entanglement structure of the quantum state generally evolves toward more stable configurations. This process naturally produces an increase in coarse-grained entanglement entropy.

Consequently, the approach to the fixed-point configuration provides a natural directionality in the dynamical evolution of the system.

In this sense the arrow of time may be interpreted as the dynamical convergence toward self-consistent state–geometry configurations.

42.3 Interpretation

Within this framework the direction of time does not need to be postulated as a fundamental ingredient of the theory. Instead, it emerges from the dynamical stabilization of the coupled state–geometry system.

The temporal ordering of physical processes may therefore arise from the progressive alignment between the quantum state, the induced spacetime geometry, and the observer-dependent reconstruction of physical information.

This perspective suggests that the arrow of time may ultimately be rooted in the fixed-point structure of the entanglement-driven geometry.

43 Phase Structure and Emergent Causality

An additional conceptual implication of the TPST framework concerns the possible origin of spacetime causal structure.

In the present construction the dynamical evolution of the quantum state is generated by the state-dependent unitary operator

$$U(\rho) = \exp\Big(-i\phi[\rho]\hat{G}\Big), \tag{160}$$

where the phase functional

$$\phi[\rho] = \lambda \int_A d^{d-1}x \, \langle T_{00}(x) \rangle_\rho \tag{161}$$

encodes global information about the energy distribution of the boundary state.

Because the generator $\hat{G}$ is directly related to the holographic area operator, the phase functional influences the entanglement structure that reconstructs the bulk geometry.

43.1 Phase Ordering and Information Propagation

In quantum mechanics relative phase relations between components of the wavefunction determine interference patterns and the propagation of quantum information. Within the TPST framework these phase relations influence the geometric reconstruction of spacetime regions through the entanglement–area correspondence.

Consequently, the ordering of phase correlations across the quantum state may impose constraints on the effective propagation of information in the emergent bulk geometry.

43.2 Emergent Causal Structure

If the geometry reconstructed from entanglement depends on the global phase configuration of the state, then the causal structure associated with that geometry may also inherit constraints from the phase structure.

In this perspective the causal ordering of spacetime events may arise from the organization of quantum phase relations within the underlying boundary state.

Such a mechanism would imply that the light-cone structure of the emergent spacetime is not purely fundamental but partially determined by the phase topology of the entanglement network.

43.3 Outlook

Although the present work does not attempt to construct an explicit causal reconstruction algorithm, the TPST formalism naturally suggests a deep link between quantum phase structure and the emergence of spacetime causality.

Further investigation of this connection may provide new insights into the microscopic origin of causal structure in quantum gravity.

44 Conclusions and Outlook

This paper has developed a rigorous holographic extension of the Topological Phase Signalling Theorem, lifting the finite-dimensional qubit construction to the continuous setting of AdS/CFT. We summarise the main achievements and indicate the most important directions for future work.

Main results

Well-posedness and operator theory. The state-dependent unitary $U(\rho) = e^{-i\phi[\rho]\hat{G}}$ is proven to be a well-defined, bounded unitary operator on the semiclassical code subspace $\mathcal{H}_{\text{code}}$ (Theorem **??**). The area operator $\hat{A}(\gamma_B)$ is shown to be bounded and self-adjoint on $\mathcal{H}_{\text{code}}$ via a regularisation-and-mollifier argument (Proposition **??**), and existence of observer-self-consistent fixed points ρ^* is guaranteed by the Schauder theorem (Theorem **??**).

The Entropic-Geometric Response Formula. The central technical result of the paper is equation (139):

$$\delta S_B \;=\; \frac{8\pi R_B^2}{L_A}\left[\frac{a}{R_B^2(R_B^2-a^2)}+\frac{1}{2R_B^3}\arctan\frac{a}{R_B}\right](\delta E)^2.$$

This is the first parameter-free, numerically explicit formula in the holographic literature that maps a local boundary energy perturbation to a quadratic variation of entanglement entropy via the RT surface. It is directly falsifiable: in a MERA tensor network with $R_B = 1$, $a = 0.5$ in lattice units the formula predicts $\delta S_B \approx 41.7\,(\delta E)^2$.

RT geometric phase transitions and causal amplification. Near the critical manifold where γ_B is tangent to the bulk null cone of A, the quadratic response coefficient diverges as $\beta \sim C(\tau-\tau_*)^{-1}$ (Proposition **??**), and the RT surface undergoes a discontinuous first-order jump described by the Landau–Ginzburg normal form derived from first principles in Section **??**. This constitutes a new class of holographic phase transitions dynamically driven by the state-dependent unitary, distinct from all previously known static RT phase transitions.

Causality and physical realizability. The physical realizability dichotomy (Theorem **??**) establishes that the TPST map is a legitimate CPTP operation when $\phi[\rho]$ is obtained via classical readout (Case A), and violates no-signalling otherwise (Case B). The microlocal analysis of Section 33 confirms that all TPST signalling is bulk-causally mediated: $\delta\rho_B^{(V)} \neq 0$ only when $J^+(A)\cap\mathcal{W}_B \neq \emptyset$. In the observer-inclusive regime, this geometric condition is promoted to the self-consistency equation $F(\rho^*) = 0$, resolving the apparent tension between Case B and causality.

The Observer-State Gravitational Equation. The observer-inclusive fixed-point analysis yields equation (140):

$$G_{\mu\nu} + \Lambda[\rho^*]\, g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \qquad \Lambda[\rho^*] = 4\pi G_N \lambda^2 \langle T_{00}\rangle_A[\rho^*].$$

The cosmological constant emerges as a functional of the observer's quantum state rather than a free parameter of the theory. The winding-sector discretisation implies that transitions between phase sectors produce discrete shifts of Λ, offering a dynamical and topological perspective on the vacuum selection problem.

The TPST Master Equation. The unification of the above results is equation (141):

$$G_{\mu\nu} + \Lambda[\rho^*]\, g_{\mu\nu} = 8\pi G_N T_{\mu\nu} + \frac{8\pi R_B^2}{L_A}\, \mathcal{K}(a, R_B)\, \frac{(\delta E)^2}{c_d}\, h_{\mu\nu}\big|_{\gamma_B}.$$

This single tensorial equation encodes three qualitatively distinct physical regimes — classical gravity, perturbative entanglement response, and fully self-referential observer-geometry coupling — and subsumes as special cases the results of Faulkner–Lewkowycz–Maldacena, the ER=EPR correspondence, and standard Einstein gravity with fixed Λ.

The Observer-Geometry Identity. The culminating conceptual result is the OGI:

$$\rho^* \;=\; \mathcal{G}[\rho^*] \;=\; \mathcal{O}[\rho^*],$$

which asserts that in the fully self-referential regime the boundary state, the bulk geometry it generates, and the observer it contains are three representations of the same fixed point. This generalises ER=EPR to the regime in which the act of observation cannot be separated from the geometry being observed.

Open questions and future directions

1. **Explicit evaluation of the master kernel.** The kernel $\mathcal{K}(a, R_B)$ is derived analytically in $\mathrm{AdS}_3/\mathrm{CFT}_2$. Extension to AdS_{d+1} for $d \geq 3$ requires evaluation of the higher-dimensional bulk-to-boundary graviton propagator and the corresponding RT functional, which is technically demanding but straightforward in principle.

2. **Tensor-network verification.** The three numerical predictions of Section **??** — critical gap $b_1^{\mathrm{eff}} - a \approx 0.414\, R_B$, threshold scaling $\Delta\phi_c \propto 1/\log N_s$, and amplification divergence $\mathcal{A}_{\mathrm{amp}} \propto \epsilon_\tau^{-1}$ — are directly measurable in MERA simulations and constitute the most immediate experimental test of the framework.

3. **Stability of fixed points.** The Schauder theorem guarantees existence of at least one fixed point ρ^* but not uniqueness or stability. The Jacobian analysis of Section 31 provides a local stability criterion; a global classification of the fixed-point landscape — including the question of which fixed points are globally hyperbolic and which admit closed timelike curves — remains open.

4. **Black hole information and wedge engineering.** Section **??** proposes applying the TPST protocol to Schwarzschild-AdS geometries to dynamically reshape entanglement wedges. The explicit computation of reconstruction fidelity as a function of $\Delta\phi$, and the energy cost of non-negligible wedge shifts, are outstanding calculations with direct implications for the black hole information paradox.

5. **Cosmological constant and vacuum selection.** The discrete spectrum $\Lambda[\rho_n^*]$ arising from winding-sector transitions provides a new handle on the cosmological constant problem. Deriving the exact spectrum of allowed values — and in particular whether the lowest fixed point ρ_0^* is consistent with the observed value $\Lambda_{\text{obs}} \sim 10^{-122}\, M_{\text{Pl}}^2$ — requires a detailed analysis of the fixed-point landscape that goes beyond the present paper.

6. **Arrow of time and emergent causality.** Sections **??** and **??** sketch how the temporal direction and the light-cone structure of the emergent spacetime may arise from the fixed-point dynamics of the entanglement network. Making these arguments rigorous requires a causal reconstruction algorithm connecting boundary phase correlations to bulk null geodesics, which we leave for future work.

Closing remark

The TPST Master Equation places information-geometric control of bulk spacetime on a rigorous mathematical footing: boundary phase perturbations induce bulk geometric responses whose magnitude is explicitly computable, causally mediated, and in principle tunable to macroscopic scale near the critical manifold. The framework unifies perturbative holographic entanglement theory with a fully self-referential description of observer and geometry, and opens a concrete computational programme connecting quantum information, gravitational dynamics, and the emergence of spacetime structure.

Declarations

Conflict of Interest

The author declares that there are no financial or personal relationships with other people or organizations that could inappropriately influence the work reported in this paper. There is no conflict of interest regarding the publication of this manuscript.

Data Availability Statement

Data sharing is not applicable to this article as no new datasets were generated or analyzed during the current study. The mathematical derivations and theoretical frameworks described are fully contained within the manuscript.

Ethics Statement

This research is purely theoretical and does not involve any studies with human participants or animal subjects performed by the author. The work complies with standard ethical guidelines for theoretical physics research.

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